ON THE EXISTENCE OF SIMPLE QUADRATURES

Seymour Haber

If S is a set of functions that are Riemann-integrable on [0, 1], then a formula

$$(1) \qquad \int_0^1 f(x)\,dx = \sum_{i=1}^{\infty} a_i f(x_i),$$

in which the x_i are distinct and the a_i and x_i are fixed independently of the function f, is called a *simple quadrature* for S if it holds for every function f in S. The functions may be complex-valued, and the a_i and x_i may be any complex numbers.

The notion of simple quadrature was introduced by Philip Davis, who investigated it in a series of papers ([1], [3], [4]; see also [2, pp. 357-358]). The term "simple" indicates the contrast with the usual numerical quadrature rules, which have the form

$$(2) \qquad \int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} a_{i,n} f(x_{i,n}).$$

There are rules of the form (2) — for example the trapezoidal rule, with $x_{i,n} = (i - 1)/(n - 1)$ and $a_{i,n} = 1/(n - 1)$ for $i \neq 1, n$; $a_{1,n} = a_{n,n} = 1/2(n - 1)$ — that are valid for all (properly) Riemann-integrable functions. In contrast, Davis showed that no rule of the form (1), with distinct x_i, is valid for all continuous functions, and he asked what classes of functions have simple quadratures. He proved [1] that there is a simple quadrature for the class of polynomials, and later [3], [4], he showed that there are some regions R in the complex plane — R containing the integration interval — such that the set of functions analytic on R has a simple quadrature. In this paper I shall construct simple quadratures for some wide classes of continuous functions; it will follow, for example, that the class of all functions continuously differentiable on [0, 1] has a simple quadrature; this extends Davis's results.

The present construction is related to a theorem of Fritz John ([5] to [7]), who found formulas of the form (1), where the x_i are not distinct, that are valid for every Riemann-integrable f. One way to obtain such formulas is as follows: We first generate a rule of the form (2) in which the sums on the right are Riemann sums: For $n = 1, 2, \cdots$, let

$$\Pi_n = (w_{n,0}, w_{n,1}, \cdots, w_{n,n}), \qquad \text{where } 0 = w_{n,0} < w_{n,1} < \cdots < w_{n,n} = 1,$$

be a partition of [0, 1] into n subintervals; and for $i = 1, 2, \cdots, n$, let $x_{n,i}$ be a point in the ith subinterval. Writing

$$a_{n,i} = w_{n,i} - w_{n,i-1} \qquad \text{and} \qquad \Delta_n = \max_i \{a_{n,i}\},$$

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