# THE ESSENTIAL RANGE OF A FUNCTION OF CLASS $H^\infty + C$

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Let $\Delta$ and T denote the open unit disc and the unit circle. Write $C = C(T)$, and denote by $H^\infty = H^\infty(T)$ the subalgebra of $L^\infty(T)$ consisting of the boundary functions (in the sense of radial limits) of bounded analytic functions on $\Delta$. E. M. Klein has asked whether a nonconstant function in $H^\infty + C$ can have a countable essential range. Negative answers have been given by Matthew Lee, who used operator-theoretic techniques, and by Colin Graham [2], who employed methods of Banach algebras. We shall prove a somewhat stronger result, using only simple function theory.

We require some notation from the theory of cluster sets; see [1] for references and an extended treatment. Briefly, for a function h defined on $\Delta$, the *cluster set* at $e^{i\theta}$ (denoted by $C(h, e^{i\theta})$) is the set of all limiting values approached by h as $z \to e^{i\theta}$ unrestrictedly in $\Delta$. Analogously, $C_\rho(h, e^{i\theta})$ is the set of cluster values of h as $z \to e^{i\theta}$ radially. Finally, we define $C_{R\setminus E}(h, e^{i\theta})$, the *boundary radial cluster set* modulo E of h at $e^{i\theta}$, by the condition that $w \in C_{R\setminus E}(h, e^{i\theta})$ provided there exist sequences $\{\theta_n\}$ and $\{w_n\}$ such that $\theta_n \to \theta$, $e^{i\theta_n} \notin E \cup \{e^{i\theta}\}$, $w_n \in C_\rho(h, e^{i\theta_n})$, and $w_n \to w$.

We need the following elementary version of Iversen's theorem (see [1, p. 98]).

LEMMA. *Let* $h \in H^\infty(\Delta)$, *and let* $E \subset T$ *have measure zero. Then* $C_{R\setminus E}(h, e^{i\theta}) \supset \partial C(h, e^{i\theta})$ *for every point of* T.

The proof follows easily from the Poisson integral representation together with the argument of [1, p. 91].

We can now prove our result.

THEOREM. *Let* $h \in H^\infty(T)$, *and suppose* h *is (essentially) discontinuous at* $e^{i\theta}$. *Then the essential range of* h *at* $e^{i\theta}$ *contains a continuum (and is therefore uncountable).*

*Proof.* Identifying h with its interior function on $\Delta$, we see that $C(h, e^{i\theta})$ is nontrivial. The boundary $\partial C(h, e^{i\theta})$ of this connected set is again nontrivial, since h is bounded. We shall now show that each point w in $\partial C(h, e^{i\theta})$ is in the essential range of h at $e^{i\theta}$.

Indeed, suppose that the point w is an exception. Then we can find an $\varepsilon > 0$ and a neighborhood U in T of $e^{i\theta}$ such that the set

$$E_1 = \{e^{i\theta}: e^{i\theta} \in U, \ |h(e^{i\theta}) - w| < \varepsilon\}$$

has measure zero. Letting $E_2$ denote the set of points on T where the radial limit of h fails to exist, and setting $E = E_1 \cup E_2$, we obtain a contradiction to the lemma.

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Received March 31, 1972.

This research was supported in part by National Science Foundation Grant GP 25084.

Michigan Math. J. 19 (1972).