APPLICATIONS OF EXTREME-POINT THEORY TO UNIVALENT FUNCTIONS

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1. INTRODUCTION

We shall show how the knowledge of the extreme points of a family of analytic functions can be an effective means of solving extremal problems. The extremal problems considered are not always linear, and our families consist primarily of various kinds of univalent functions. The most striking of our new results deal with close-to-convex functions. We indicate how our approach is useful in a variety of situations, and how it affords a systematic treatment of several classical results in the theory of univalent functions.

Let $\mathscr{A}$ denote the set of all functions analytic in the unit disk $\Delta = \{z: |z| < 1\}$. Then $\mathscr{A}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. Let S denote the subset of $\mathscr{A}$ consisting of the univalent functions that satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Also, let St, K, C, and R denote the subsets of S consisting of starlike, convex, close-to-convex, and real mappings, respectively, so that, for example, $f \in \mathrm{St}$ or $f \in K$ if the domain $f(\Delta)$ is starlike (with respect to the origin) or convex, and $f \in R$ if $f(z)$ is real when z is real $(-1 < z < 1)$. The family C was introduced in [10] by W. Kaplan, and it can be described in terms of a geometric mapping property. Analytically, a function f is in C if there exist a function g and a complex number a such that $ag \in \mathrm{St}$ and $\Re\{zf'(z)/g(z)\} > 0$ for $|z| < 1$. We recall the inclusion relations $K \subset \mathrm{St} \subset C$.

We shall discuss some observations made in [3] by L. Brickman, D. R. Wilken, and this author. Let $\mathfrak{H}B$ denote the closed convex hull of the set B; also, let $\mathfrak{E}(\mathfrak{H}B)$ denote the extreme points of $\mathfrak{H}B$. Each of the four families St, K, C, and R is locally uniformly bounded, because S has this property. In fact, each family is even compact. This implies that $\mathfrak{H}\mathrm{St}$, $\mathfrak{H}K$, $\mathfrak{H}C$, and $\mathfrak{H}R$ are compact. Consequently (see [5, p. 440])

$$\mathfrak{E}(\mathfrak{H}\mathrm{St}) \subset \mathrm{St}, \quad \mathfrak{E}(\mathfrak{H}K) \subset K, \quad \mathfrak{E}(\mathfrak{H}C) \subset C, \quad \mathfrak{E}(\mathfrak{H}R) \subset R.$$

These four sets of extreme points were completely determined in [3], as follows:

$$\mathfrak{E}(\mathfrak{H}\mathrm{St}) = \{f: f(z) = z/(1 - \varepsilon z)^2 \quad (|\varepsilon| = 1)\},$$

$$\mathfrak{E}(\mathfrak{H}K) = \{f: f(z) = z/(1 - \varepsilon z) \quad (|\varepsilon| = 1)\},$$

$$\mathfrak{E}(\mathfrak{H}C) = \left\{f: f(z) = \left[z - \frac{1}{2}(\varepsilon + \delta)z^2\right]/[1 - \delta z]^2 \quad (|\varepsilon| = |\delta| = 1,\ \varepsilon \neq \delta)\right\},$$

$$\mathfrak{E}(\mathfrak{H}R) = \{f: f(z) = z/(1 + bz + z^2) \quad (-2 \leq b \leq 2)\}.$$

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