HOLOMORPHIC IDEMPOTENTS AND COMMON FIXED POINTS ON THE 2-DISK

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1. INTRODUCTION

Several authors have studied the question whether two functions that map a set into itself and commute under composition must have a common fixed point. In [4], J. P. Huneke shows that continuous, commuting functions on the unit interval need not have a common fixed point. H. H. Glover and Huneke [5] have discussed the general problem of spaces without the common-fixed-point property for commuting selfmaps. In [7], A. L. Shields showed that if $\mathscr{F}$ is a commuting family of functions holomorphic in the unit disk Δ in $\mathbb{C}$, continuous in $\overline{\Delta}$, and mapping $\overline{\Delta}$ into $\overline{\Delta}$, then the elements of $\mathscr{F}$ have a common fixed point.

In this note, we prove an analogous result for the 2-disk. To do this, we first obtain a characterization of the holomorphic idempotents of the 2-disk into itself.

We wish to thank Henry Glover for his continued interest in this material.

2. HOLOMORPHIC IDEMPOTENTS ON Δ^2

The *2-disk* is the set $\Delta \times \Delta = \Delta^2$ in $\mathbb{C} \times \mathbb{C}$. For each pair (z_1, z_2) in $\mathbb{C} \times \mathbb{C}$, let $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$. By a *disk in* $\mathbb{C} \times \mathbb{C}$ we shall mean a set of the form $\{(\rho_1 z, \rho_2 z): z \in \Delta\}$, where $\|(\rho_1, \rho_2)\| \neq 0$. We shall need the following form of Schwarz's lemma in Δ^2.

LEMMA. *If* $F: \Delta^2 \to \Delta$ *is holomorphic, with* $F(0, 0) = 0$ *and* $|F| \leq M$, *then* $|F(z_1, z_2)| \leq M\|(z_1, z_2)\|$.

Moreover, if there exists a pair (z_1^*, z_2^*) *in* $\Delta^2 - \{(0, 0)\}$ *such that* $|F(z_1^*, z_2^*)| = M\|(z_1^*, z_2^*)\|$, *then, with the notation* $\rho_i \|(z_1^*, z_2^*)\| = z_i^*$ $(i = 1, 2)$, F *is linear on the disk* $\{(\rho_1 z, \rho_2 z): z \in \Delta\}$.

Proof. Writing each pair (z_1, z_2) in Δ^2 as (zw_1, zw_2), where $\|(w_1, w_2)\| = 1$ and $|z| = \|(z_1, z_2)\|$, we see, by applying Schwarz's lemma to the function $G(z) = F(zw_1, zw_2)$, that for $\|(z_1, z_2)\| = r$,

$$|F(z_1, z_2)| \leq \max_{|z|=r} |F(zw_1, zw_2)| \leq r \max_{|z|<1} |F(zw_1, zw_2)| \leq Mr.$$

Now, if (z_1^*, z_2^*) is a point such that $\|(z_1^*, z_2^*)\| = r$ $(0 < r < 1)$ and $|F(z_1^*, z_2^*)| = Mr$, then, setting $\rho_i = z_i^*/r$ for $i = 1, 2$ and applying Schwarz's lemma to the function $G(z) = F(\rho_1 z, \rho_2 z)$, we see that $G(z) = \eta z$, where $|\eta| = 1$. From the double power series for F, we find that there are constants A_1 and A_2 such that $F(\rho_1 z, \rho_2 z) = A_1 \rho_1 z + A_2 \rho_2 z$; this yields the result.

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