FINITE GROUPS IN WHICH ANY TWO PRIMARY SUBGROUPS OF THE SAME ORDER ARE CONJUGATE

Fletcher Gross

1. INTRODUCTION

Define a class $\mathscr{C}$ of finite groups as follows: the group G belongs to $\mathscr{C}$ provided that whenever H and K are subgroups of G of the same order, then H and K are conjugate in G. A. Machí [10] showed that if $G \in \mathscr{C}$ and some Sylow 2-subgroup of G is either an elementary abelian group of order 4 or a quaternion group of order 8, then A_4, the alternating group on 4 letters, is involved in G. The hypothesis $G \in \mathscr{C}$ seems so strong that it is natural to expect a stronger conclusion than Machí's result. One of the main results of the present paper is that if $G \in \mathscr{C}$, then $G/O_{2'}(G)$ is isomorphic to one of the following groups: a cyclic 2-group, A_5, $SL_2(5)$, $PSL_2(8)$, $P\Gamma L_2(32)$, A_4, $SL_2(3)$, or specific solvable groups of orders 56, 168, or 4,960. Thus the only simple nonabelian groups in $\mathscr{C}$ are A_5 and $PSL_2(8)$.

If p is a prime, the class $\mathscr{C}_p$ consists of the finite groups G with the property that whenever H and K are p-subgroups of the same order in G, then H and K are conjugate in G. Finally, let $\mathscr{D}$ consist of the groups that belong to $\mathscr{C}_p$ for every prime p. Clearly, $\mathscr{C} \subseteq \mathscr{D}$; but the reverse is not true. In Theorem 1, we list all the possibilities for $G/O_{2'}(G)$ if $G \in \mathscr{D}$. This immediately leads to the classification of groups belonging to $\mathscr{C}$.

2. NOTATION AND PRELIMINARY RESULTS

All groups considered in this paper are assumed to be finite. We use repeatedly the fact that the classes $\mathscr{C}$, $\mathscr{C}_p$, and $\mathscr{D}$ are closed under the operation of taking factor groups. J_1 denotes the simple group of order 175,560, discovered by Z. Janko [9]. If p is a prime and n is a positive integer, then the groups $R(p^n)$, $S(p^n)$, and $T(p^n)$ are defined as follows: Let V be the additive group of the field $GF(p^n)$, and let λ be a primitive $(p^n - 1)$th root of unity in $GF(p^n)$. Let A and B be the automorphisms of V defined by

$$vA = \lambda v \quad \text{and} \quad vB = v^p \qquad \text{for } v \in V.$$

Then A and B generate a group $T(p^n)$ of order $n(p^n - 1)$. The semidirect product of V and the cyclic group generated by A is denoted by $R(p^n)$, while the semidirect product of V and $T(p^n)$ is denoted by $S(p^n)$. The orders of $R(p^n)$ and $S(p^n)$ are $p^n(p^n - 1)$ and $np^n(p^n - 1)$, respectively. All other notation is as in D. Gorenstein's book [5].

LEMMA 1. *Suppose* $G \in \mathscr{C}_p$ *and* P *is a Sylow* p*-subgroup of* G. *Then one of the following is true:*

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