SETS OF CONSTANT DISTANCE FROM A PLANAR SET

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Let A be a compact subset of the Euclidean plane R^2. For each $\varepsilon > 0$, define

$$\partial_\varepsilon(A) = \varepsilon\text{-boundary of } A = \{x \in R^2 \colon \|x - A\| = \varepsilon\},$$

where $\|x - A\| = \inf_{a \in A} \|x - a\|$ is the distance from x to A. I shall prove that

(i) $\partial_\varepsilon(A)$ is the union of a finite collection of simple closed curves minus the union of their interiors, and therefore

(ii) each component of $\partial_\varepsilon(A)$ is locally connected, which implies that

(iii) for all but a countable number of ε, each component of $\partial_\varepsilon(A)$ is a point, a simple arc, or a simple closed curve.

The key idea for (i) works in R^n, but (ii) and (iii) require restriction to the plane.

First consider the case where ε is large compared with the diameter $\sup_{\alpha,\alpha' \in A} \|\alpha - \alpha'\|$ of A.

LEMMA 1. *Let* A *have diameter* δ, *where* $\delta < \varepsilon$, *and suppose that* A *contains the origin* 0. *Then* $\partial_\varepsilon(A)$ *is an* (n - 1)*-sphere. In fact, there exists a homeomorphism* H *of* R^n *upon itself such that*

$$\text{(i)} \quad \begin{cases} \dfrac{H(x)}{\|H(x)\|} = \dfrac{x}{\|x\|} & (x \neq 0), \\ H(0) = 0, \end{cases}$$

(ii) H *carries the unit* (n - 1)*-sphere onto* $\partial_\varepsilon(A)$,

(iii) H *carries the interior of the unit* (n - 1)*-sphere onto*

$$V_\varepsilon(A) = \{x \in A \mid \|x - A\| < \varepsilon\}.$$

Proof. For each point σ on the unit sphere S^{n-1}, let Λ_σ denote the half-line $\Lambda_\sigma = \{x \in R^n \mid x/\|x\| = \sigma\}$. If x and y are two points of Λ_σ and $\delta \leq \|x\| < \|y\|$, then $\|x - A\| < \|y - A\|$. To see this, let T_σ be the (n - 1)-hyperplane normal to Λ_σ at $\delta\sigma \in \Lambda_\sigma$. By elementary geometry, each point on the other side of T_σ from x is closer to x than to y. This includes all points of A. Now let σ be a fixed point of S^{n-1}, and consider the function $t \to \|t\sigma - A\|$ $(0 < t < \infty)$. We have just observed that $\|t\sigma - A\|$ is strictly increasing with t as long as $\delta \leq t$. For $t < \sigma$,

$$\|t\sigma - A\| \leq \|t\sigma - 0\| = \|t\sigma\| < \delta < \varepsilon,$$

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