SOME APPLICATIONS OF A THEOREM OF W. M. SCHMIDT

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1. INTRODUCTION

A module $\mathfrak{M}$ in an algebraic number field K is called *degenerate* if $\mathfrak{M}$ has a submodule $\mathfrak{N}$ such that for some $\alpha \in K$, $\alpha\mathfrak{N}$ is a full module in some subfield K' of K, where K' is neither the field of rational numbers nor an imaginary quadratic field. In [3, Satz 2], W. M. Schmidt obtained the following remarkable generalization of Thue's theorem:

Let K *be an algebraic number field of degree at least* 3, *and let* $\alpha_1, \cdots, \alpha_n$ *be linearly independent elements of* K. *If the module generated by* $\alpha_1, \cdots, \alpha_n$ *over the integers* Z *is nondegenerate, then the equation*

$$N(\alpha_1 x_1 + \cdots + \alpha_n x_n) = C,$$

where N *denotes the norm from* K *to the rational field* Q *and where* C *is a rational number, has only finitely many solutions in integers* $x_1, \cdots, x_n$.

In the present paper, we shall make certain applications of Schmidt's theorem. Among other things, we shall generalize and improve certain theorems of Siegel and of Nagell. Our results are as follows.

THEOREM 1. *Let* h *be a positive integer, and let* θ *be an algebraic number of degree* $n > 2h$. *Let* N *be the norm from* $Q(\theta)$ *to* Q. *Then for each rational number* C, *the equation*

$$N(x_0 + \theta x_1 + \cdots + \theta^h x_h) = C$$

has only a finite number of integral solutions $x_0, x_1, \cdots, x_h$.

Siegel [4] had proved a result of this type, with the stronger hypothesis that

$$n > h^2\left(\frac{n}{s+1} + s\right), \quad \text{where } 2s = \sqrt{4n+1} - 1.$$

We note that the present hypothesis $n > 2h$ is in a certain sense best possible. For if $n = 2h$, let α be a real quadratic irrational with the property that $\theta = \sqrt[h]{\alpha}$ is of degree n. Then the equation $N(x_0 + x_h \alpha) = N(x_0 + x_h \theta^h) = C$ does have infinitely many solutions x_0, x_h for suitable values of C, and hence the equation $N(x_0 + \theta x_1 + \cdots + \theta^h x_h) = C$ has infinitely many solutions.

THEOREM 2. *Let* n *be an integer greater than* 1 *that is not divisible by* 2, 3, *or* 5. *Let* ξ *be a primitive* n*th root of unity, and let* r, s, t *be rational integers with* $0 \leq r < s < t < n$. *Let* N *be the norm from* $Q(\xi)$ *to* Q. *Then, for each rational constant* C, *the equation*

$$N(\xi^r x + \xi^s y + \xi^t z) = C$$

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