AN EXAMPLE OF TORRIGIANI RELATED TO MULTIPLE FOURIER SERIES

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A function of n variables is in W_p^1 if it is absolutely continuous in each variable for almost all values of the other variables and if both it and its partial derivatives are in L_p; it is in V_p if it is of bounded variation in each variable for almost all values of the other variables and if the n variation functions are in L_p, each as a function of the other $n - 1$ variables. For each $p \geq 1$, $V_p \supset W_p^1$.

For the case $n = 2$, L. Cesari [1, Theorem 3, p. 290] showed that if $f \in V_1$, then the rectangular sums of its double Fourier series converge almost everywhere. Subsequently, L. Tonelli [4, p. 325] suggested a new proof of this. For the case $n = 3$, Cesari [2] also showed that if $p > 1$ and $f \in V_p$, then the rectangular sums of the triple Fourier series of f converge almost everywhere. The issue whether Cesari's result holds for $p = 1$ and $n = 3$ remains unresolved. Evidence toward a negative answer is furnished by an example of G. Torrigiani [5], for $n = 3$, of a function $f \in V_1$ possessing a certain property nowhere. For $n = 2$, each $f \in V_1$ has this property almost everywhere. It is used by Tonelli [4] to show that for $n = 2$ the double Fourier series of each $f \in V_1$ converges almost everywhere. The example given by Torrigiani is long and involved. In view of the revived interest in this topic, we feel that it is worthwhile to give a short, simple discussion, which, however, is based on Torrigiani's idea.

Let Q_3 be the unit cube, and Q_2 the unit square. For each interval $[a, b] \subset [0, 1]$, each function f, and each point $(x, y) \in Q_2$, let $V_z(f; x, y; [a, b])$ denote the variation of f as a function of z on the interval [a, b] with x and y fixed. Since we shall be dealing only with absolutely continuous functions, we need not worry about jumps at the endpoints. Tonelli's condition for functions of two variables at a point (x_0, y_0) is that for each $\varepsilon > 0$ there exists a $\lambda > 0$ such that $0 < \delta < \lambda$ implies

$$\frac{1}{2\delta}\int_{y_0-\delta}^{y_0+\delta} V_x(f; y_0; [x_0, x_0+\lambda])\,dy < \varepsilon .$$

He showed that if $f \in V_1$, the condition is satisfied almost everywhere, and this implies that the Fourier series of f converges almost everywhere.

A function g is an *S-function* (after S. Saks, see [3]) if it is summable and

$$\limsup_{h,k\to 0} \frac{1}{4hk}\int_{x-h}^{x+h}\int_{y-k}^{y+k} g(u, v)\,du\,dv = +\infty ,$$

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