THE DISTANCE TO VERTICAL ASYMPTOTES FOR SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

L. E. Bobisud

A function $y(t)$ is said to have a vertical asymptote at a if either

$$\lim_{t \to a-} |y(t)| = \infty \quad \text{or} \quad \lim_{t \to a+} |y(t)| = \infty .$$

Solutions of nonlinear differential equations of the form

$$y'' = p(t) f(y) \tag{1}$$

may have vertical asymptotes; for example, $\tan(t - \alpha)$ is the solution of

$$y'' = 2y(1 + y^2) \tag{2}$$

satisfying the conditions $y(\alpha) = 0$ and $y'(\alpha) = 1$. This solution is defined to the right of α up to $\alpha + \pi/2$, where it has a vertical asymptote. Similarly, $\sec(t - \alpha)$ satisfies the equation

$$y'' = 2y^3 - y$$

and has a vertical asymptote at $\alpha + \pi/2$.

If for the solution $y(t; \alpha) = y(t)$ of (1) satisfying the conditions $y(\alpha) = a$ and $y'(\alpha) = b$ we denote by $t(\alpha)$ the location of the vertical asymptote of y to the right of α ($t(\alpha) = \infty$ if y is defined on $[\alpha, \infty)$), then it is meaningful to discuss the asymptotic behavior of $t(\alpha) - \alpha$ as $\alpha \to \infty$. This is the analogue of the question of the asymptotic distribution of zeros for oscillatory solutions of differential equations. Theorem 1 below answers this question under certain assumptions on p and f.

Theorems 2 and 3 give implicit lower bounds on the distance to vertical asymptotes of solutions of certain equations of the form (1). S. B. Eliason [1] has obtained such lower bounds under restrictions on f different from ours.

1. ASYMPTOTIC BEHAVIOR OF $t(\alpha) - \alpha$

Concerning (1), we assume *that* f *is continuous on* $(-\infty, \infty)$, *that* p *is positive and continuously differentiable on* $[0, \infty)$, *and that* $p'(t)\, p(t)^{-3/2} \to 0$ *as* $t \to \infty$. We shall deal only with the case where $y(\alpha) > 0$ and $y'(\alpha) \geq 0$; similar statements and proofs apply to the case where $y(\alpha) \leq 0$ and $y'(\alpha) \leq 0$, also to the problem of the distance to the vertical asymptote to the left of α. We consider then the solution $y(t) = y(t; \alpha)$ (assumed unique) of (1) satisfying the conditions

Received November 3, 1971.

Michigan Math. J. 19 (1972).