TWO-SLIT MAPPINGS AND THE MARX CONJECTURE

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1. INTRODUCTION

Let S^* denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ that map the unit disk $|z| < 1$ conformally onto a domain starlike with respect to the origin. Let $k(z) = z(1 - z)^{-2}$ denote the Koebe function. In 1932, A. Marx [5] proved that for each $f \in S^*$, the function $f(z)/z$ is subordinate to $k(z)/z$, and he conjectured that $f'(z)$ is subordinate to $k'(z)$. In other words, the Marx conjecture asserts that if

$$K_r = \{k'(z): |z| \leq r\} \qquad (0 \leq r < 1)$$

and

$$M_r = \{f'(r): f \in S^*\},$$

then $M_r = K_r$ for each r $(0 \leq r < 1)$.

Marx proved this conjecture for all $r \leq \sin \pi/8 = 0.382 \cdots$. R. M. Robinson [7], [8] improved the constant to $(5 - \sqrt{17})/2 = 0.438 \cdots$, and later to 0.6. P. L. Duren [1] made a further improvement to $0.736 \cdots$. R. McLaughlin [6] obtained the same constant with a different method. However, J. A. Hummel [2] has recently constructed a counterexample, which shows that the Marx conjecture is false for all sufficiently large r. Hummel's example is a mapping

$$f(z) = \frac{z}{(1 - ze^{is})^b (1 - ze^{it})^{2-b}} \qquad (0 \leq b \leq 2;\ 0 \leq s, t \leq 2\pi) \tag{1}$$

of the disk onto the plane slit along two rays. The construction is based on a continuity argument that gives a counterexample for sufficiently small values of b. On the other hand, Hummel shows that, for each $r < 1$, every point on the boundary of the Marx region M_r corresponds to a two-slit mapping of the form (1). Thus the Marx problem is actually equivalent to an analogous question concerning two-slit mappings.

We shall prove that for two-slit mappings with equal weights b and $2 - b$, the Marx conjecture is true. This result is stated more precisely below. Recall that if F and G are analytic functions in the unit disk, then F is said to be *subordinate* to G if $F(z) = G(\omega(z))$ for some analytic function ω with $|\omega(z)| \leq |z|$.

THEOREM 1. *If* f *is a function of the form* (1) *with* $b = 1$, *then* f' *is subordinate to* k'.

The proof depends on a study of the valence of the functions k' and f'. In particular, we find that $\sqrt{k'}$ is univalent. In Section 4, we show that $\log k'$ is starlike.

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