LIMITS OF NILPOTENT AND QUASINILPOTENT OPERATORS

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The structure of the set of nilpotent operators on an infinite-dimensional Hilbert space is still incompletely described. Many natural questions, suggested by finite-dimensional results, remain to be answered. One such question, raised by P. R. Halmos [2, Question 7], asks for a description of the closure of the nilpotent operators in the uniform operator topology. In this paper, we show that most self-adjoint operators are not limits of nilpotent or quasinilpotent operators, but that many interesting (and not quasinilpotent) weighted shifts are. The results suggest that a simple characterization of the closure of the nilpotent operators may be difficult to discover.

Throughout this paper, H will be a separable complex Hilbert space, usually infinite-dimensional, and an operator will be a bounded linear transformation on H. We shall denote by $\mathscr{N}$ the set of nilpotent operators on H (all operators S with $S^k = 0$ for some k), by $\mathscr{Q}$ the set of quasinilpotent operators (all S with $r(S) = \lim \|S^k\|^{1/k} = 0$), and by $\mathscr{N}^-$ and $\mathscr{Q}^-$ the respective closures in the uniform operator topology. If H is finite-dimensional, then $\mathscr{N}^- = \mathscr{N} = \mathscr{Q} = \mathscr{Q}^-$; in general, $\mathscr{N}$ is properly contained in $\mathscr{Q}$. It is still unknown whether $\mathscr{Q} \subset \mathscr{N}^-$. We follow the notation of [1, p. 37] for the various parts of the spectrum of an operator: Λ will denote the spectrum, Π_0 the point spectrum, Π the approximate point spectrum, and Γ the compression spectrum.

1. SPECTRAL PROPERTIES OF $\mathscr{N}^-$ AND $\mathscr{Q}^-$

Since the quasinilpotent operators are precisely those whose spectrum is the single point $\{0\}$, the problem of characterizing $\mathscr{Q}^-$ is that of describing which operators can be approximated by operators with spectrum $\{0\}$. Clearly, the spectral radius is discontinuous near such operators. It is thus natural to expect that spectral properties will give partial and incomplete information about $\mathscr{Q}^-$. Note that it suffices to investigate operators of norm 1, since $\mathscr{N}$ and $\mathscr{Q}$ are closed under multiplication by scalars.

PROPOSITION 1. *If* T *is bounded below by* ε, *then*

$$d(T, \mathscr{Q}) = \inf\{\|T - S\| : S \in \mathscr{Q}\} \geq \varepsilon .$$

Proof. If S is quasinilpotent, then $0 \in \Pi(S)$. Thus there exists a sequence of vectors $\{x_n\}$ with $\|x_n\| = 1$ and $\|Sx_n\| \to 0$. Hence

$$\|(T - S)x_n\| \geq \|Tx_n\| - \|Sx_n\| \geq \varepsilon - \|Sx_n\| \to \varepsilon ,$$

so that $\|T - S\| \geq \varepsilon$.

COROLLARY 1. *If* T *is invertible, then* $T \notin \mathscr{Q}^-$. *Equivalently, if* $T \in \mathscr{Q}^-$, *then* $0 \in \Lambda(T)$.

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