HYPERINVARIANT SUBSPACES FOR OPERATORS ON THE SPACE OF COMPLEX SEQUENCES

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Let (s) denote the space of all complex sequences (functions on the positive integers) with the seminorms

$$\|f\|_n = \max_{1 \le j \le n} |f(j)| \qquad (n = 1, 2, \cdots).$$

By an operator on (s) we mean a continuous linear transformation of (s) into itself; by a subspace of (s) we mean a closed vector subspace. A subspace is said to be *invariant* for an operator if it is mapped into itself by the operator, and *hyperinvariant* (see [1]) if it is invariant for every operator commuting with the given operator. In this note we show that to each operator on (s) that is not a scalar multiple of the identity operator, there corresponds a proper hyperinvariant subspace. This answers a question raised in [3].

Notation. By $\mathbb{C}$ we denote the complex field, and by (s_0) the space of all sequences of complex numbers that have only finitely many nonzero elements. Thus (s_0) is a vector space of dimension $\aleph_0$ over $\mathbb{C}$.

There is a duality between (s) and (s_0):

$$(1) \qquad (f, p) = \sum f(n)\, p(n) \qquad (f \in (s),\ p \in (s_0)).$$

Each p induces a continuous linear functional on (s), and every continuous linear functional has this form. Further, each f induces an algebraic linear functional on (s_0), and every algebraic linear functional has this form. The space (s) and the space (s_0) with its strong dual topology, that is, the topology of uniform convergence on bounded subsets of (s), are dual spaces. Every linear transformation on (s_0) is continuous, and every linear subspace in (s_0) is closed.

If S is a vector subspace of (s_0), then $S^\perp$ denotes the annihilator of S in (s). This is always a closed subspace, and it is proper if and only if S is proper.

THEOREM 1. *Every operator on* (s) *that is not a scalar multiple of the identity has a proper hyperinvariant subspace.*

Proof. Let U be an operator on (s). Because of the duality between (s) and (s_0), it will be sufficient to show that the adjoint transformation U^* on (s_0) has a proper hyperinvariant subspace; the annihilator of this subspace will be the desired subspace for U. The following lemma and its corollary will complete the proof.

LEMMA 1. *Every algebraic linear transformation on* (s_0) *has nonempty spectrum.*

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