BOUNDED FUNCTIONS WITH ONE-SIDED SPECTRAL GAPS

Harold S. Shapiro

1. INTRODUCTION

It is a well-known theorem of Sidon that if the sequence $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ of Fourier coefficients of a bounded, measurable, 2π-periodic function f defined on the real line $\mathbb{R}$ has Hadamard lacunarity, then $\sum |\hat{f}(n)| < \infty$ (for terminology and references, see [9, vol. I, p. 247]). In particular, the gap condition implies that f is *continuous* (after correction on a set of measure zero); moreover, it is known that lacunarity hypotheses weaker than those in Sidon's theorem imply continuity (H. P. Rosenthal [3]). If we assume only that $\{\hat{f}(n)\}$ has *infinitely many* Hadamard gaps, then continuity of f is not guaranteed, but certain kinds of discontinuous behavior are ruled out. For instance, f cannot have a jump discontinuity; this is a consequence of well-known facts about conjugate Fourier series (it is not difficult to deduce it from Theorem 8.13 in Chapter 2 of [9]; I am grateful to Professor Zygmund, who supplied me with this reference).

In results of the type just described, a "gap" in the sequence of Fourier coefficients means the vanishing of both the sine and cosine coefficients, for a certain block of indices; that is, in terms of the sequence $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$, a gap is understood to be *symmetric* about $n = 0$. The main point of this paper (Corollary to Theorem 2) is that *one-sided* gaps, that is, sufficiently long blocks of consecutive zeros in the sequence $\{\hat{f}(n)\}_{n=-\infty}^{-1}$, are incompatible with jump discontinuities. More generally, one-sided gaps force a kind of matching behavior, in a sense of averages, on the values of a function in left- and right-hand neighborhoods of each point. Results of the latter kind do not seem to be explicitly known, even for symmetric gaps; at any rate, we do not know of any studies along these lines.

Observe that no *one-sided* gap condition can force so strong a regularity as continuity upon a bounded function: even the most drastic conceivable one-sided gap condition, namely that $\hat{f}(n) = 0$ for all $n < 0$, means only that f is the radial boundary function of a bounded analytic function, which needn't be continuous. But such a function cannot have a jump discontinuity, by virtue of a classical theorem of Pringsheim and Lindelöf; generalizations of this, involving matching average behavior, were noted in [5]. (For other generalizations of the no-jump theorem, see [7], [8].)

The present paper can be viewed as a sequel to [5], and it overlaps that paper slightly; however, here we employ a variant of the method in [5] that enables us to handle functions having one-sided gaps, not merely boundary values in H^{∞}. Because the generalization does not complicate matters, we formulate our results for functions on $\mathbb{R}^n$ (as in [5]). Specialization to $n = 1$ and 2π-periodic functions yields results on traditional Fourier series.

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