ON A PARTITION THEOREM OF SYLVESTER

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1. INTRODUCTION

G. E. Andrews [1] recently gave an analytic proof of a classical theorem (a generalization of Euler's partition theorem $\prod_{n=1}^{\infty}(1+q^n) = 1/\prod_{n=1}^{\infty}(1-q^{2n-1})$) due to Sylvester:

THEOREM. *Let* $A_k(n)$ *denote the number of partitions of* n *into odd parts (repetition allowed) with exactly* k *distinct parts. Let* $B_k(n)$ *denote the number of partitions of* n *into mutually distinct parts such that* k *maximal sequences of consecutive integers appear in each partition. Then* $A_k(n) = B_k(n)$.

In his paper, Andrews asked for a direct proof of the identity

$$(1)\quad F(a, q) = 1 + \sum_{k,n} B_k(n) a^k q^n = \sum_{r=1}^{\infty} q^{r(r-1)/2} \frac{(1+(a-1)q)\cdots(1+(a-1)q^r)}{(1-q)\cdots(1-q^{r-1})},$$

and we now give such a proof.

Andrews also gave a proof of the following identity of V. A. Lebesgue (see [3, p. 42]):

$$(2)\qquad \sum_{r=0}^{\infty} q^{r(r+1)/2} \frac{(1+\beta q)(1+\beta q^2)\cdots(1+\beta q^r)}{(1-q)(1-q^2)\cdots(1-q^r)} = \prod_{r=1}^{\infty}\left(\frac{1+\beta q^{2r}}{1-q^{2r-1}}\right).$$

We derive this identity by proving the more general identity

$$(3)\qquad \sum_{m=0}^{\infty} q^{m(m+1)/2} \frac{(z)_m}{(q)_m}\alpha^m = (z)_\infty(-\alpha q)_\infty \sum_{n=0}^{\infty} \frac{z^n}{(q)_n(-\alpha q)_n},$$

where

$$(a)_n \equiv (a; q)_n \equiv (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{and} \quad (a)_\infty \equiv (a; q)_\infty \equiv \lim_{n\to\infty} (a; q)_n .$$

This identity becomes obvious if we expand both sides of (3) in powers of z, α, and q, and compare the coefficients of similar terms. We point out that the identity (2) is a special case of the q-analogue of Kummer's theorem [5] (let $b \to \infty$ in Daum's identity). The identity (3) is a special case of a theorem of E. Heine [6, p. 106]. To see this relation, replace α with α/τ in equation (1.6) of [2], and then set $\gamma = 0$ and let $\tau \to 0$. An identity more general than (3) is also found in the Notebooks of S. Ramanujan [7, p. 194]:

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