CONVEX HYPERSURFACES

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1. INTRODUCTION

Concerning the relation between the topology and curvature K of a Riemannian manifold, it is known that

(i) an Hadamard manifold (a complete, simply-connected manifold with $K \leq 0$) is diffeomorphic to Euclidean space,

(ii) if a simply-connected, complete manifold is 1/4-pinched (that is, if $1/4 < K \leq 1$), it is homeomorphic to a sphere, and

(iii) a complete open manifold of positive curvature and of dimension at least 5 must be diffeomorphic to Euclidean space.

In this paper, we investigate hypersurfaces that are embedded in an Hadamard manifold or in a 1/4-pinched complete Riemannian manifold and satisfy the semiconvexity condition defined in Section 2. We prove the following two theorems.

THEOREM A. *Let* M^n $(n \neq 4, 5)$ *be a simply-connected,* 1/4-*pinched, complete Riemannian manifold, and let* N^{n-1} *be a simply-connected, semiconvex, compact hypersurface embedded in* M. *Then* N *is homeomorphic to* S^{n-1}.

THEOREM B. *Every semiconvex, compact hypersurface embedded in an Hadamard manifold is diffeomorphic to a sphere.*

The proofs use a modification of an argument due to Hadamard. The restriction on n in Theorem A arises from the application of a theorem in [5, p. 264]. Both theorems generalize the results of F. J. Flaherty [3], [4].

2. CONVEX HYPERSURFACES AND STAR-SHAPED SETS

Let M^n be a Riemannian manifold diffeomorphic either to R^n or to S^n. By a well-known separation theorem, each compact, connected embedded hypersurface N divides M into two components. On the other hand, suppose z is a fixed unit normal vector field on N in M, and let r denote the injectivity radius of N in M [5]; define two subsets of M - N as

$$A = \{\text{Exp}\, tz: 0 < t \leq r\} \quad \text{and} \quad B = \{\text{Exp}\, t(-z): 0 < t \leq r\}. \tag{1}$$

Both A and B are the images of connected sets under the continuous map Exp. Consequently, both are connected, and $A \cup B = M - N$. However, by the separation theorem, M - N has exactly two components, and therefore A and B must be the components of M - N.

Next, recall the second fundamental form L_z, defined by the equation

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