STABILIZATION OF SELF-EQUIVALENCES OF THE PSEUDOPROJECTIVE SPACES

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1. INTRODUCTION

For a space X with basepoint, let $\mathcal{E}(X)$ denote the group of homotopy classes of homotopy equivalences of X into itself, the group operation being composition. We refer to $\mathcal{E}(X)$ as the self-equivalence group of X. The operation of suspending one homotopy equivalence to obtain another determines a sequence of homomorphisms

$$\mathcal{E}(X) \xrightarrow{\Sigma} \mathcal{E}(\Sigma X) \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \mathcal{E}(\Sigma^n X) \xrightarrow{\Sigma} \cdots$$

connecting the self-equivalence groups of the iterated suspensions of X. When X is a finite CW complex, this sequence stabilizes at some stage $\mathcal{E}(\Sigma^n X)$ $(0 \leq n \leq \dim X)$, in that it consists of isomorphisms thereafter.

We describe this stabilization process in the case where X is the pseudoprojective plane of order q, denoted by P_q^1. As a starting point we take P. Olum's description [6] of the rather rich structure of $\mathcal{E}(P_q^1)$. Let Γ_q denote the quotient of the integral polynomial ring Z[x] modulo the ideal generated by $1 + x + \cdots + x^{q-1}$, and let E_q denote the group whose elements are the units of Γ_q and whose multiplication $\circ$ is defined by the formula

$$\left\{\sum n_i x^i\right\} \circ \left\{\sum m_i x^i\right\} = \left\{\sum n_i x^i\right\} \left\{\sum m_i x^{is}\right\},$$

where $s = \sum n_i \pmod q$ is called the augmentation of $\left\{\sum n_i x^i\right\}$.

THEOREM 1 ([6, Theorems 3.4 and 3.5, and Remark 3.6]). *The self-equivalence group* $\mathcal{E}(P_q^1)$ *of the pseudoprojective plane* P_q^1 *is isomorphic to the group* E_q. *Moreover,* E_q *is isomorphic to the semidirect product* $U_q^1 \times_\theta Z_q^*$ *of the group* U_q^1 *(of units of* Γ_q *of augmentation* 1) *and the multiplicative group* Z_q^* *(of reduced residues modulo* q) *whose operators* $\theta\colon Z_q^* \to \mathrm{Aut}\, U_q^1$ *are given by the relation* $\theta(s)\left(\left\{\sum n_i x^i\right\}\right) = \left\{\sum n_i x^{is}\right\}$.

Since the pseudoprojective plane P_q^1 admits a two-dimensional cellular decomposition, namely, $S^1 \cup_q e^2$, the stabilization process takes at most two steps; hence the relevant suspensions are the pseudoprojective spaces $P_q^2 = S^2 \cup_q e^3$ and $P_q^3 = S^3 \cup_q e^4$. Our description of the stabilization process is summarized by the following two theorems.

THEOREM 2. *The self-equivalence group* $\mathcal{E}(P_q^2)$ *is isomorphic to the semidirect product* $Z_q \times_\phi Z_q^*$ *of the cyclic group* Z_q *of order* q *and the group* Z_q^* *whose operators* $\phi\colon Z_q^* \to \mathrm{Aut}\, Z_q$ *are given by the canonical isomorphism*

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