THE RELATIVE GROWTH OF SUBORDINATE FUNCTIONS

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1. INTRODUCTION

Suppose the functions f and F are regular in the unit disk K and vanish at the origin. The function f is said to be subordinate to F in K (in symbols: $f \prec F$) if there exists a function ω regular in K with the properties that $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in K$), and $f(z) \equiv F(\omega(z))$. In all sufficiently small disks $K_r = \{z\colon |z| < r\}$, functionals of r and f are in general dominated by corresponding functionals of r and F, whenever $f \prec F$. Many authors have studied the problem of determining the largest disk where such a domination takes place. For example, G. M. Golusin [4] proved the following result. Let a(r) and A(r) denote the areas of the Riemann surfaces $f(K_r)$ and $F(K_r)$, respectively. Then

$$a(r) \leq A(r) \qquad (0 \leq r \leq 1/\sqrt{2}),$$

provided $f \prec F$. E. Reich was the first to investigate a more general problem. He obtained estimates of the ratio a(r)/A(r) in the whole unit disk under the assumption that $f \prec F$, and he proved the inequality [7]

$$a(r)/A(r) \leq mr^{2m-2} \qquad \left(\frac{m-1}{m} \leq r^2 \leq \frac{m}{m+1};\ m = 1, 2, \cdots\right),$$

which implies Golusin's result in the case where $m = 1$.

In this paper, we study the least upper bound of another ratio. The authors thank Professor J. G. Krzyż for suggesting this problem.

Let A_n ($n = 1, 2, \cdots$) denote the class of functions f regular in K such that

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \qquad (a_n \geq 0).$$

Let S denote the class of functions regular and univalent in K, subject to the usual normalizations. Suppose S_0 is some fixed subclass of S, and suppose that for each η ($|\eta| < 1$), the function $\eta^{-1} f(\eta z)$ belongs to S_0 whenever $f \in S_0$. Define

$$\kappa(r, n, S_0) = \sup\{|f(z)/F(z)|\colon f \in A_n,\ F \in S_0,\ f \prec F,\ |z| = r\}$$

(n is a positive integer, and $0 < r < 1$). We are able to determine $\kappa(r, n, S^*)$ and $\kappa(r, n, S_c)$, where S^* denotes the class of functions starlike with respect to the origin and S_c denotes the class of convex functions.

Let B_n ($n = 1, 2, \cdots$) denote the class of functions ω regular in K and satisfying the conditions

$$\omega(z) = \alpha_n z^n + \alpha_{n+1} z^{n+1} + \cdots \qquad (\alpha_n \geq 0)$$

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