ON A CLASS OF SCHLICHT FUNCTIONS

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Let S be the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

that are analytic and schlicht in $|z| < 1$. In 1934, O. Dvořák [1] made the interesting observation that if $f \in S$ and

$$\Re \sqrt{\frac{f(z)}{z}} > \frac{1}{2} \quad (|z| < 1), \tag{1}$$

then $|a_n| \leq n$ $(n = 2, 3, \cdots)$. The proof simply uses the fact that

$$\sqrt{\frac{f(z)}{z}} - \frac{1}{2} = \frac{1}{2} + c_1 z + c_2 z^2 + \cdots$$

has positive real part, so that $|c_n| \leq 1$ $(n = 1, 2, \cdots)$. Dvořák [1] further showed that every function $f \in S$ satisfies (1) in the disk $|z| < \rho$, where

$$\rho \log \frac{1+\rho}{1-\rho} = 2\,.$$

A calculation shows that $0.833 < \rho < 0.834$. Recently, Dvořák [2] claimed to show that (1) holds in a disk $|z| < r_0$, where $0.90 < r_0 < 0.91$. He later [3] claimed an improvement to $0.98 < r_0 < 0.99$.

Unfortunately, however, these last two estimates are incorrect. In the present note, we show that the best possible radius is $R = 0.835 \cdots$. In other words, for every $f \in S$, the inequality (1) holds for every z in $|z| < R$; but for each z in $|z| > R$, there is some $f \in S$ for which (1) fails to hold. Our procedure allows the computation of R to any desired accuracy. It is curious that although Dvořák derived the constant ρ by what appears to be crude estimation, the sharp constant R is only slightly larger.

For $f \in S$, G. M. Goluzin [5] used Loewner's differential equation to establish the sharp estimate

$$\left|\arg \frac{f(z)}{z}\right| \leq \log \frac{1+r}{1-r} \quad (r = |z| < 1)\,.$$

Thus

$$\Re \sqrt{\frac{f(z)}{z}} > 0 \quad \text{for all } f \in S$$

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