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A NOTE ON MAPPING CYLINDERS

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THEOREM. *Let X and Y be topological spaces, and let $f: X \rightarrow Y$ be a homotopy equivalence. Then $X \times \{0\}$ is a strong deformation retract of the mapping cylinder $M_f = X \times I \cup_f Y$.*

This theorem is known in homotopy theory, but the proof normally comes out of the context of cofibration (see [1, Corollary 3.7], for example). I feel that a direct proof is desirable, since it is simple and since in special cases, for example, for fiber-homotopy equivalences or for equivariant-homotopy equivalences, one can verify directly that a corresponding theorem holds. Thus one is not forced to re-appraise a whole theory.

The first theorem in this direction was proved by R. Fox in 1943 (see [2, Theorem 3.7]).

Proof. Let $g: Y \rightarrow X$ be a homotopy inverse of f , and let $K: X \times I \rightarrow X$ be such that $K(x, 0) = x$ and $K(x, 1) = g \circ f(x)$. Consider the set

$$\{S \mid S: X \times I \rightarrow X \text{ with } S(x, 0) = x, S(x, 1) = g \circ f(x)\}.$$

We denote the set of homotopy classes of these maps by $\mathcal{P}(X \times I, X; 1_X, g \circ f)$. Also, we form the set of homotopy classes $\mathcal{P}(Y \times I, Y; 1_Y, f \circ g)$. The map $f: X \rightarrow Y$ induces the maps

$$\mathcal{P}(X \times I, X; 1_X, g \circ f) \xrightarrow{f^*} \mathcal{P}(X \times I, Y; f, f \circ g \circ f)$$

and

$$\mathcal{P}(Y \times I, Y; 1_Y, f \circ g) \xrightarrow{f_*} \mathcal{P}(X \times I, Y; f, f \circ g \circ f);$$

f^* and f_* are equivalences of sets, since f is a homotopy equivalence (for the proof, one needs the proposition that canonical change of "base points" induces equivalences of sets). Therefore there must exist a map $L: Y \times I \rightarrow Y$ such that $L(y, 0) = y$ and $L(y, 1) = f \circ g(y)$, and such that $f^*[L] = f_*[K]$. Consequently, there exists a homotopy $M: X \times I \times I \rightarrow Y$ such that

$$M(x, t, 0) = f \circ K(x, t), \quad M(x, 0, s) = f(x)$$

and

$$M(x, t, 1) = L(f(x), t), \quad M(x, 1, s) = f \circ g \circ f(x)$$

for $x \in X$ and $t, s \in I$.

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