# DERIVATIVES OF SINGULAR INNER FUNCTIONS

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Let U denote the open unit disc $\{z: |z| < 1\}$, and let T denote the unit circle $\{z: |z| = 1\}$. For $0 < p < \infty$, the Hardy class $H^p$ consists of all functions f analytic in U for which

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

is finite. An analytic function f is said to be of bounded characteristic ($f \in N$) in case

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

is finite, and to be in class $B^p$ $(0 < p < 1)$ if

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})| \, (1 - r)^{1/p-2} \, dr \, d\theta$$

is finite. It is well known that $H^p \subset N$ [4, p. 16], and that $H^p \subset B^p$ for $0 < p < 1$ [5, p. 415].

A singular inner function is a function of the form

$$S(z; \mu) = \exp\left( - \int \frac{e^{it} + z}{e^{it} - z} \, d\mu(e^{it}) \right),$$

where $\mu$ is a positive measure on T, singular with respect to Lebesgue measure on T (see Chapter 5 of [6] for details). Recently, much attention has been given to the factorization and boundary properties of functions with derivatives in $H^p$ (see [1], [2], and [3], for instance). In [2], J. G. Caughran and A. L. Shields have raised the problem of finding conditions on the singular measure $\mu$ sufficient to insure that $S'(z; \mu) \in H^p$ for some $p > 0$. Is it possible that $S'(z; \mu) \in H^{1/2}$? Does there exist a singular inner function $S(z; \mu)$ such that $S'(z; \mu) \in H^p$ and the distribution function of $\mu$ is continuous? Theorems 1 and 4 of this paper give conditions on $\mu$ sufficient to insure that $S'(z; \mu)$ belongs to $H^p$ or N, and they answer the latter question in the affirmative. Theorem 2 shows that in case $S'(z; \mu) \in H^{1/2}$, the support $\sigma(S)$ of $\mu$ must be perfect and may not be a Carleson set. Recall that a Carleson set is a closed subset of T that has measure zero and whose complement is the union of open arcs of lengths $\varepsilon_n$, where $\sum \varepsilon_n \log 1/\varepsilon_n < \infty$. Finally, we use Theorem 4 to give an example of a singular inner function whose derivative is in $H^p$ $(p < 1/4)$ and whose support is a perfect non-Carleson set.

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