THE HARDY CLASS OF A SPIRAL-LIKE FUNCTION

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1. INTRODUCTION

A univalent function f analytic on the open unit disk Δ is said to be *spiral-like* of order σ ($|\sigma| < \pi/2$) if it is normalized ($f(0) = 0$ and $f'(0) = 1$), and if in addition it satisfies the condition

$$\Re[e^{i\sigma} z f'(z)/f(z)] > 0 \qquad (z \in \Delta).$$

(Spiral-like functions were considered by L. Špaček [5].) *For each spiral-like function* f, *we shall determine, by studying the region* $f(\Delta)$, *the Hardy classes* H_p *to which* f *belongs.* This is the object of Theorem 1, which is stated in Section 2 and proved in Section 3. The proof will use the notion of the Hardy number of a region, which was defined and studied in [1]. Theorem 1 enables us to draw some conclusions concerning the growth of the maximum modulus and the Taylor coefficients of spiral-like functions (Section 4).

2. PRELIMINARIES

Let Ω be a region (that is, a connected nonempty open set in the finite complex plane) that contains the point $z = 0$. We shall say that Ω is *spiral-like* of order σ ($|\sigma| < \pi/2$) if, whenever $z_0 \in \Omega$, the spiral $\{z_0 \exp(te^{-i\sigma}): t \leq 0\}$ is also contained in Ω. We note that if $\sigma = 0$, then Ω is starlike with respect to the point $z = 0$.

The relationship between spiral-like functions and spiral-like regions is indicated by the following lemma.

LEMMA 1. *Let* f *be a normalized univalent function analytic on the unit disk* Δ. *Then* f *is spiral-like of order* σ *if and only if* $f(\Delta)$ *is spiral-like of order* σ.

A proof of the special case $\sigma = 0$ is given by W. Hayman in [3, pp. 14-15]. The proof of the general case is similar.

Let Ω be a region, and let I_Ω denote the identity map on Ω. We recall from [1] that the *Hardy number* of Ω is defined by the condition

$$h(\Omega) = \sup\{p \geq 0: |I_\Omega|^p \text{ possesses a harmonic majorant}\}. \tag{1}$$

The most significant property of $h(\Omega)$ is that if f is analytic on Δ, $f(\Delta) \subseteq \Omega$, and $h(\Omega) > 0$, then f belongs to each Hardy class H_p ($0 < p < h(\Omega)$). We shall also use the facts that

(i) if $\Omega_1 \subseteq \Omega_2$, then $h(\Omega_2) \leq h(\Omega_1)$,

(ii) if $\Omega_2 = \{az + b: z \in \Omega_1\}$ ($a \neq 0$), then $h(\Omega_2) = h(\Omega_1)$.

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