ON THE MARX CONJECTURE FOR A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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We let $G(z) \prec H(z)$ ($|z| < R$) mean that $G(z)$ is subordinate to $H(z)$ in $|z| < R$ in the sense that $G(z)$ and $H(z)$ are regular in the disk $|z| < R$, and for each fixed $r < R$, the image of the disk $|z| \le r$ under $G(z)$ is contained in its image under $H(z)$. Let S_α^* $(0 \le \alpha < 1)$ denote the class of functions that are starlike of order α in the open unit disk $E = \{z: |z| < 1\}$; that is, $f(z)$ belongs to S_α^* if and only if $f(z)$ is regular in E, $f(0) = 0$, $f'(0) = 1$, and

$$\Re\{zf'(z)/f(z)\} \ge \alpha$$

for every z in E.

In 1932, A. Marx [3] conjectured that for every $f(z) \in S_0^*$, the function $f'(z)$ is subordinate to $k'(z)$ in E, where $k(z) = z/(1-z)^2$ is the Koebe function. B. Pinchuk [5] and R. McLaughlin [4] have studied the corresponding conjecture for the classes S_α^*, namely that $f'(z) \prec k_\alpha'(z)$ in E for every $f(z) \in S_\alpha^*$, where

$$k_\alpha(z) = z/(1-z)^{2-2\alpha}.$$

For each $\alpha \in [0, 1)$, McLaughlin [4] has determined a number r_α $(0 < r_\alpha < 1)$ such that the Marx conjecture for S_α^* holds in the disk $|z| \le r_\alpha$. The constant $r_0 = 0.736\cdots$ in [4] had been discovered earlier by P. L. Duren [1]. For $\alpha = 1/2$, it was shown that $r_{1/2} = 0.81046\cdots$ [4]. In this note, we consider a class of close-to-convex functions that contains $S_{1/2}^*$ as a proper subclass, and we show that for every $f(z)$ in this class, the relation $f'(z) \prec k_{1/2}'(z)$ holds in the entire disk E.

For $0 \le \alpha < 1$ and $0 \le \beta < 1$, we say that $f(z) \in \mathscr{K}(\alpha, \beta)$ if and only if

(i) $f(z)$ is regular in E, $f(0) = 0$, $f'(0) = 1$, and

(ii) for some $g(z) \in S_\beta^*$,

$$\Re\{zf'(z)/g(z)\} \ge \alpha \tag{1}$$

for every z in E. We note that $\mathscr{K}(\alpha, \beta)$ is a subclass of the class of close-to-convex functions of order α and type β introduced by R. J. Libera [2]. Instead of condition (ii), Libera required that (1) hold for some $g(z)$ such that $ag(z) \in S_\beta^*$ for some complex number a of modulus 1 [2, Definition (1.2)]. The class S_α^* is the subset of functions $f(z) \in \mathscr{K}(\alpha, \alpha)$ such that $g(z) = f(z)$ in (1).

We shall need the Herglotz representations for the classes S_β^* and $\mathscr{K}(\alpha, \beta)$. Let I denote the class of nondecreasing functions with total variation 1 on the interval $[0, 2\pi]$. It is well known that $g \in S_\beta^*$ if and only if

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