ON COMMUTATORS IN IDEALS OF COMPACT OPERATORS

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1. INTRODUCTION

Let $\mathscr{H}$ be an infinite-dimensional, separable Hilbert space, and denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded, linear operators on $\mathscr{H}$. All operators to which we refer will be bounded and linear. We denote by $\mathscr{K}$ the (two-sided) ideal in $\mathscr{L}(\mathscr{H})$ consisting of all compact operators, and we recall that $\mathscr{K}$ is a C^*-algebra (without identity). The ideal $\mathscr{K}$ is the only ideal in $\mathscr{L}(\mathscr{H})$ that is closed in the operator norm, but $\mathscr{K}$ contains a whole chain of other ideals—the Schatten p-ideals. For our purposes, it will be convenient to employ Dixmier's method [3, p. 296] of defining the *Schatten* p-*classes* $\mathscr{C}_p$, where p is any positive number. To do this, one first defines the *trace class* $\mathscr{C}_1$ as follows. Suppose $K \in \mathscr{K}$, and let $K = UP$ be the canonical polar decomposition of K. Then P is a positive compact operator, and thus is unitarily equivalent to a diagonal matrix—say (δ_i). One says that $K \in \mathscr{C}_1$ if the sequence $\{\delta_i\}$ belongs to the Banach space (ℓ_1). The trace *norm* of an operator $K \in \mathscr{C}_1$ is defined by $\|K\|_1 = \|\{\delta_i\}\|_1$, where the norm on the right is the norm on the Banach space (ℓ_1). The set of all operators $A^{1/p}$, where A runs over all positive operators in $\mathscr{C}_1$, is the positive part of an ideal in $\mathscr{K}$ [3, Proposition 1, p. 296], and we denote this ideal by $\mathscr{C}_p$. It is easy to see that if $K \in \mathscr{C}_p$ and K has a canonical polar decomposition $K = UP$, then $P^p \in \mathscr{C}_1$, and we define the *Schatten* p-*norm* of K by $\|K\|_p = \|P^p\|_1^{1/p}$. It is known that $\mathscr{C}_p$ is a Banach *-algebra under the Schatten p-norm [6]. Furthermore, it is not hard to see that for every pair p, q of positive numbers p and q,

$$\mathscr{C}_p \cdot \mathscr{C}_q = \mathscr{C}_r, \tag{1}$$

where $r^{-1} = p^{-1} + q^{-1}$, and where the left-hand side represents the set of all finite sums of the form $\sum A_i B_i$ $(A_i \in \mathscr{C}_p,\ B_i \in \mathscr{C}_q)$. The ideals $\mathscr{C}_1$ and $\mathscr{C}_2$ are more important than the other Schatten p-ideals. The trace class $\mathscr{C}_1$ may be characterized as the set of all compact operators K with the property that the matrix of K with respect to every orthonormal basis of $\mathscr{H}$ has absolutely summable trace (whose value is independent of the orthonormal basis). We shall denote the trace on $\mathscr{C}_1$ by $\operatorname{tr}(\cdot)$, and we recall that it has all the usual properties of a trace [6]. The ideal $\mathscr{C}_2$ is called the *Hilbert-Schmidt class*, and it may be characterized as the set of operators K in $\mathscr{L}(\mathscr{H})$ such that some (and therefore every) matrix for K has square-summable entries [6]. We refer the reader to [6] for more detail concerning the Schatten p-classes.

If $\mathscr{M}$ is any ideal in $\mathscr{L}(\mathscr{H})$, we denote by $C(\mathscr{M})$ the set of all commutators of elements of $\mathscr{M}$. In other words, $C(\mathscr{M})$ consists of all operators A (necessarily in $\mathscr{M}$) such that there exist operators $B, C \in \mathscr{M}$ with $A = BC - CB$. We also denote the linear span of $C(\mathscr{M})$ by $[\mathscr{M}, \mathscr{M}]$. In other words, $[\mathscr{M}, \mathscr{M}]$ consists of all finite sums of elements from $C(\mathscr{M})$. In the case $\mathscr{M} = \mathscr{L}(\mathscr{H})$, the identification of

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