SEMINORMAL OPERATORS

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A bounded linear operator T on a Hilbert space is called a *seminormal* operator if $T^*T - TT^* = D \geq 0$ or $D \leq 0$. Several authors, especially C. R. Putnam, J. G. Stampfli, and S. K. Berberian, have determined conditions that assure the normality of a seminormal operator. Let $\mathscr{B}(H)$ denote the algebra of all bounded operators on a Hilbert space H, and $\mathscr{K}$ the ideal of all compact operators. Let $\hat{T}$ be the image of T in $\mathscr{B}(H)/\mathscr{K}$, under the quotient map, and let $\sigma(\hat{T})$ be the spectrum of $\hat{T}$ in the C^*-algebra $\mathscr{B}(H)/\mathscr{K}$. In Section 1, we show that T is normal whenever T is a seminormal operator and $\sigma(\hat{T})$ consists of certain arcs and a countable set. This will imply that T is normal if it is seminormal and the spectrum of a compact perturbation of T lies on certain arcs plus a countable set. These results extend some results obtained by T. Yoshino [13], the author [4], and Stampfli [8] to [11].

In Section 2, we use the results of Section 1 to obtain several theorems giving algebraic conditions under which T is normal. If T is a seminormal operator such that $I - T^*T$ is compact and $i(T - \lambda I) = 0$ (i is the Fredholm index) for some λ with $|\lambda| \leq \|T\|^{-1}$, then T is normal. From this we derive conditions on the strong asymptotic behavior of T and T^* that imply the normality of a seminormal operator T. For a seminormal contraction for which the rank of $I - T^*T$ is finite, we present necessary and sufficient conditions on the asymptotic behavior of T and T^* that imply normality.

1. SPECTRAL CONDITIONS

The *Weyl spectrum* $\omega(T)$ of T is defined as $\bigcap \sigma(A + K)$, where the intersection is taken over all K in $\mathscr{K}$ [3].

Our results are based on the relations among $\sigma(T)$, $\omega(T)$, and $\sigma(\hat{T})$. Whenever H is infinite-dimensional, then $\sigma(\hat{T}) \subset \omega(T) \subset \sigma(T)$, and each of these sets is a nonempty, compact subset of the plane. An operator is said to satisfy Weyl's theorem if $\omega(T) = \sigma(T) - \pi_{00}(T)$, where $\pi_{00}(T)$ is the set of isolated eigenvalues of finite multiplicity. L. A. Coburn [3] has shown that hyponormal operators (that is, operators for which $T^*T - TT^* \geq 0$) satisfy Weyl's theorem, and S. K. Berberian has shown that seminormal operators satisfy Weyl's theorem [1].

Recall that an operator is called a *semi-Fredholm* [*Fredholm*] operator if its range R(T) is closed and its null space N(T) is finite-dimensional [if N(T) and $R(T)^\perp$ are finite-dimensional]. The semi-Fredholm [Fredholm] operators constitute an open set in $\mathscr{B}(H)$. We shall denote the set of Fredholm operators by $\mathscr{F}$. If T is a semi-Fredholm operator, the *index* of T is defined to be

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