ON SPECIAL *-REGULAR RINGS

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1. A ring R is *regular* if for every $a \in R$ there exists an $x \in R$ such that $axa = a$. A regular ring with an involution * is called **-regular* if $xx^* = 0$ implies $x = 0$. In this note, we study *-regular rings with unit 1 that possess some additional properties. Before specifying these properties, let us state some known facts about regular rings.

Two idempotents e and f of a ring R are *equivalent* (notation: $e \sim f$) if $e = xy$ and $f = yx$, where $x \in eRf$ and $y \in fRe$. For each element a of a *-regular ring R, we have the relations $aR = eR$ and $Ra = Rf$, where e and f are uniquely determined projections (self-adjoint idempotents), called the *left* and the *right projection* of a. Further, there exists a uniquely determined element $\bar{a}$ - the *relative inverse* of a - such that $a\bar{a} = e$ and $\bar{a}a = f$. The left and the right projections of any element a are equivalent [1].

A *-regular ring is *complete* (alternate terminology: a *regular Baer *-ring*) if the lattice of its projections is complete. Let $\{e_\alpha\}$ be any set of projections. By LUB e_α and GLB e_α we shall denote the least upper bound and the greatest lower bound of the set $\{e_\alpha\}$. A complete *-regular ring R is finite; that is, if a projection $e \in R$ is equivalent to 1, then $e = 1$. The *central cover* of a projection $e \in R$ is the smallest central projection g in R for which $ge = e$.

A regular ring is *abelian* if all its idempotents are central. An idempotent $e \in R$ is *abelian* if the subring eRe is abelian. A *-regular ring R is of *type* I if it has an abelian projection with central cover 1, and of *type* II if it does not possess nonzero abelian projections.

Every complete *-regular ring is a direct sum of a ring of type I and a ring of type II (see [1], [3]). A ring of type I is *n-homogeneous* (or of *type* I_n) if there exists a set of n mutually orthogonal equivalent abelian projections whose sum is 1. Every complete *-regular ring of type I is a special subdirect sum of homogeneous rings. The proof of this theorem can be found in [1], [2], [3]. Here we shall give a proof of the structure theorem based on the following known properties of complete *-regular rings of type I: (a) abelian projections with the same central cover are equivalent, and (b) if g is the central cover of a projection f, then there exists an abelian projection $e \leq f$ with central cover g.

First we show that in every complete *-regular ring of type I there exist a sequence of orthogonal abelian projections e_n and a decreasing sequence of central projections g_n with the following properties: The central cover of e_n is g_n. If $h_n = g_n - g_{n+1}$ and $h_n \neq 0$, then $h_n e_1, \cdots, h_n e_n$ are orthogonal equivalent abelian projections with the sum h_n. Thus the subring Rh_n is n-homogeneous.

The construction of these sequences is by induction on n. Since R is of type I, there exists an abelian projection e with central cover 1. We put $e_1 = e$ and $g_1 = 1$. Suppose now that the projections $e_1, \cdots, e_n$ and $g_1, \cdots, g_n$ with the required

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