A CLASSIFICATION OF HYPERELLIPTIC RIEMANN SURFACES WITH AUTOMORPHISMS BY MEANS OF CHARACTERISTIC RIEMANN MATRICES

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It has been shown [5] that a hyperelliptic Riemann surface S of even genus g has an automorphism (conformal self-homeomorphism) σ of order 2 other than the interchange ι of sheets if and only if S has a Riemann matrix of the form

$$\frac{1}{2}\begin{pmatrix} \hat{M} & I \\ I & -\tilde{M}^{-1} \end{pmatrix} \quad \text{or, equivalently,} \quad \frac{1}{2}\begin{pmatrix} \tilde{M}+\hat{M} & \tilde{M}-\hat{M} \\ \tilde{M}-\hat{M} & \tilde{M}+\hat{M} \end{pmatrix},$$

where all the entries are submatrices of order g/2, and where I is the multiplicative identity matrix. Furthermore, $\tilde{M}$ and $\hat{M}$ are Riemann matrices for the quotient surfaces S/σ and $S/\iota\sigma$, respectively, which are elliptic or hyperelliptic; in the latter case, the natural projections map the hyperelliptic branch points (Weierstrass points) of S over the Riemann sphere P to the hyperelliptic branch points of the respective quotient surfaces over P. A similar result holds for odd genus. The object of this paper is to complete the classification of hyperelliptic Riemann surfaces with automorphisms by means of characteristic Riemann matrices.

Let S be a compact Riemann surface of genus $g > 0$. A set of (independent) one-cycles (a_i, b_i) $(i = 1, \cdots, g)$ satisfying the conditions

$$\delta(a_i, b_j) = \delta_{ij} \quad \text{and} \quad \delta(a_i, a_j) = 0 = \delta(b_i, b_j),$$

where δ is the bilinear, skew-symmetric intersection number, is called a set of *retrosections* for S, and the corresponding homology basis is said to be *canonical*. If $\omega_1, \cdots, \omega_g$ form a basis for the holomorphic differentials on S, then the $g \times 2g$ matrix

$$(A\ B) = \left(\left(\int_{a_j} \omega_i\right)\left(\int_{b_j} \omega_i\right)\right)$$

is called a *period matrix* for S. By a change of basis for the holomorphic differentials, the matrix A can be reduced to the multiplicative identity (the new basis is said to be *normalized* with respect to (a_i, b_i)), and then B becomes $A^{-1}B$, which is symmetric, has positive-definite imaginary part, and is called the *Riemann matrix for* S *with respect to* (a_i, b_i). Torelli's theorem says that if the Riemann matrix for a surface S with respect to (a_i, b_i) is the same as the Riemann matrix for a surface S' with respect to (a_i', b_i'), then some conformal homeomorphism from S onto S' takes either a_i to a_i' and b_i to b_i', or a_i to $-a_i'$ and b_i to $-b_i'$ (in the sense that homologous cycles are identified; see [4, pp. 27-28] and [3]). If S' (and therefore S) is hyperelliptic, then conformality of one map implies conformality of the other,

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