AMBIGUOUS POINTS OF HOLOMORPHIC FUNCTIONS OF SLOW GROWTH

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1. INTRODUCTION

Let f denote a holomorphic function in the open unit disc D. An *arc at* $e^{i\theta}$ is a curve $J \subset D$ such that $J \cup \{e^{i\theta}\}$ is a Jordan arc. The complex number a ($a = \infty$ is admitted) is an *asymptotic value* of f at $e^{i\theta}$ if there exists an arc at $e^{i\theta}$ on which f has the limit a at $e^{i\theta}$. Let $\Gamma(f, e^{i\theta})$ denote the set of asymptotic values of f at $e^{i\theta}$. If $\Gamma(f, e^{i\theta})$ contains at least two values, then $e^{i\theta}$ is called an *ambiguous point* of f.

It follows from the work of E. Lindelöf [8] that an f omitting two finite values has no ambiguous points (for a generalization of this result, see [7]). However, a result of W. Gross [5] can be used to show that even if f omits only one finite value, $\Gamma(f, 1)$ may nevertheless contain every complex number. By F. Bagemihl's ambiguous-point theorem [1], the set of ambiguous points of any f is at most countable. Ambiguous points of various classes of functions have been studied in [2], [3], [4], and [10].

Suppose $a \in \Gamma(f, e^{i\theta})$, and let J be an arc at $e^{i\theta}$ on which f has the limit a at $e^{i\theta}$. For each $\varepsilon > 0$, let $G(a, J, \varepsilon)$ denote the component of $\{z: |f(z) - a| < \varepsilon\}$ (of $\{z: |f(z)| > \varepsilon^{-1}\}$ if $a = \infty$) such that $G(a, J, \varepsilon) \cap J$ contains an arc at $e^{i\theta}$. The collection $\{G(a, J, \varepsilon): \varepsilon > 0\}$ is called the *tract* (or asymptotic tract) *of* f *at* $e^{i\theta}$ associated with the asymptotic value a and determined by J. Let

$$T_a = \{G(a, J, \varepsilon): \varepsilon > 0\} \quad \text{and} \quad T_b = \{G(b, J', \varepsilon): \varepsilon > 0\}$$

be tracts of f at $e^{i\theta}$. Then T_a and T_b are *distinct* if there exists an $\varepsilon > 0$ such that $G(a, J, \varepsilon) \cap G(a, J', \varepsilon) = \emptyset$. Note that the tracts are automatically distinct if $a \neq b$. However, more than one tract may be associated with an element $a \in \Gamma(f, e^{i\theta})$.

Let $n_*(f, e^{i\theta})$ ($n_\infty(f, e^{i\theta})$) denote the cardinal number of the set of tracts of f at $e^{i\theta}$ associated with finite (infinite) asymptotic values. For $0 < r < 1$, let $M(f, r)$ denote the maximum modulus of f on the circle $\{z: |z| = r\}$, and for $x > 0$, let $\log^+ x = \max(\log x, 0)$. G. R. MacLane [10, p. 54] has obtained the following results:

(A) *if* $\int_0^1 \log^+ M(f, r)\,dr < \infty$, *then*

$$n_*(f, e^{i\theta}) \leq 1 \quad \textit{and} \quad n_\infty(f, e^{i\theta}) \leq 2 \qquad \text{for each } \theta\,;$$

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