BRANCHED COVERINGS

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1. INTRODUCTION

In this paper we consider branched coverings of compact manifolds. A map f of a compact n-manifold M onto an n-manifold N is a *branched covering* if $f^{-1}f(B_f) = B_f$ and $f \mid (M - B_f)$ is a finite-to-one covering map. Here B_f denotes the set of points of M at which f is not a local homeomorphism. If $f \mid f^{-1}f(B_f)$ is a homeomorphism, the branched covering is a Montgomery-Samelson fibering with zero codimension, and we call it an M-S *covering*. If $f \mid f^{-1}f(B_f)$ is a covering map, we call it a *singular covering*. If $f \mid (M - B_f)$ is a regular covering, we call f a *regular branched covering*. In Section 2, we prove some theorems about general branched coverings. In Section 3, we construct a special homology theory and use it to investigate the structure of the branch set for M-S coverings. In Section 4 we study branched coverings by spheres, and in Section 5 we study branched coverings onto spheres. Section 6 contains some examples and remarks involving smooth branched coverings. We call $f: M \to N$ *smooth* if both M and N are n-manifolds with a C^m structure and f is C^m. We call f *simplicial* if M and N can be triangulated so that f is simplicial with respect to the triangulations. For a survey of problems related to this paper, see [9].

2. BRANCHED COVERINGS

PROPOSITION 1. *Let* $f: X \to Y$ *be an open map from the compact, path-connected and locally path-connected space* X *to the connected and locally simply connected Hausdorff space* Y. *Suppose that* $q = \min\{\text{card } f^{-1}(y): y \in Y\}$ *is finite. Then* $f_{\#}\pi(X)$ *has at most* q *cosets in* $\pi(Y)$, *where* π *denotes the fundamental group and* $f_{\#}$ *denotes the homomorphism induced by* f.

Proof. Suppose that $f_{\#}\pi(X)$ has p cosets in $\pi(Y)$ and that $p > q$. Since f(X) is open and compact, hence closed, in Y, the mapping f is onto Y. Therefore Y is path-connected and locally path-connected. Let $g: Z \to Y$ be the covering map corresponding to $f_{\#}\pi(X)$, and let $h: X \to Z$ be the lift of f. The map h is open, because f is open and g is a local homeomorphism. It follows that h is onto Z. Since g is a p-to-1 map, we infer that, for each y in Y,

$$\text{card } f^{-1}(y) = \text{card } h^{-1}g^{-1}(y) \geq \text{card } g^{-1}(y) = p > q,$$

contrary to the choice of q.

COROLLARY 1.1. *If* $f: M \to N$ *is a singular covering,* $\dim B_f \leq n - 2$, *and* $f \mid B_f$ *is* p-to-1, *then* $f_{\#}\pi(M)$ *has at most* p *cosets in* $\pi(N)$.

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