ON CERTAIN LINKS IN 3-MANIFOLDS

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1. INTRODUCTION

A. H. Wallace [7] has shown that each closed, orientable 3-manifold can be transformed into the 3-sphere by removing a finite number of solid tori (a regular neighborhood of a compact, connected graph with Euler characteristic $1 - n$) and sewing them back in an appropriate manner. In particular, this can be done with any homotopy 3-sphere. In Theorem 2 we show that if the complement of the solid tori has free fundamental group and if the tori lie "nicely" with respect to some Heegaard splitting of the manifold, then the manifold is the connected sum of lens spaces, and therefore, if it is simply connected, it is the 3-sphere.

2. PRELIMINARIES

All spaces considered will be polyhedra, and all maps will be piecewise linear. We shall denote the n-sphere by S^n. It is known [1] that every closed, orientable 3-manifold can be represented as $H \cup H'$, where H and H' are solid tori of some genus n and $H \cap H' = \partial H = \partial H' = T$ is an orientable surface of genus n. Such a representation, denoted by $(H, H'; T)$, is a *Heegaard splitting of genus* n for the manifold. A manifold that possesses a Heegaard splitting of genus 1 is called a *lens space*. Note that this class of manifolds consists of the classical lens spaces along with the two spaces S^3 and $S^2 \times S^1$. The symbol $\mathrm{cl}(A)$ will denote the closure of A.

Let H be a solid torus of genus n, and let $D_1, \cdots, D_k$ $(k \leq n)$ be pairwise disjoint disks properly embedded in H, that is, let $D_i \cap \partial H = \partial D_i$ for each i. Let $J_1, \cdots, J_k$ be pairwise disjoint, simple closed curves in ∂H such that $J_i \cap D_i$ is a point for each i and $J_i \cap D_j = \emptyset$ for $i \neq j$. Then $\{J_1, \cdots, J_k\}$ is called a *set of* k *canonical longitudes for* H.

Let M and N be oriented, closed 3-manifolds, and let $B \subset M$ and $E \subset N$ be 3-cells. Let $h: \partial B \to \partial E$ be an orientation-reversing homeomorphism. Then the orientable, closed 3-manifold $(M - \mathrm{int}(B)) \cup_h (N - \mathrm{int}(E))$ is called the *connected sum of* M *and* N, and it is denoted by $M \# N$. Note that this is independent of the choices of B, E, and h.

3. THE THEOREMS

THEOREM 1. *Let* M *be a closed, orientable* 3-*manifold, and let* $J_1 \cup \cdots \cup J_n = J$ *be a link in* M. *Suppose* $\pi_1(M - J)$ *is a free group. Then* M *is homeomorphic to the connected sum of a homotopy* 3-*sphere with a finite number of lens spaces.*

Proof. First note that if N is a solid torus of genus 1 and E is a 3-cell such that $N \cap E = \partial N \cap \partial E = A$ is an annulus, then $N \cup E$ is a lens space with an open

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