ON EQUATIONS IN FREE GROUPS

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1. INTRODUCTION

It is well known that if elements $u_1, \cdots, u_n$ in a free group satisfy some nontrivial relation $w(u_1, \cdots, u_n) = 1$, then the rank of the free subgroup generated by $u_1, \cdots, u_n$ is at most $n - 1$. We are interested in conditions on w under which such a subgroup can in fact have rank $n - 1$. We obtain a necessary and sufficient condition (see Theorem 3), namely, that $w = w(x_1, \cdots, x_n)$ lie in the normal closure of some element from a free basis for the free group F freely generated by $x_1, \cdots, x_n$. Unfortunately, this is not entirely satisfactory, since no general method is known for deciding whether a word w meets the criterion. However, for special classes of w, we succeed in making this condition more explicit.

One special case of our problem has received some attention. If elements a, b, and c of a free group satisfy a relation $a^m b^n c^p = 1$, where $|m|, |n|, |p| \geq 2$, then the rank of the group generated by a, b, and c is at most 1. This was proved for $|m| = |n| = |p| = 2$ by R. C. Lyndon [6], for $|m| = |n| = |p| \geq 2$ by E. Schenkman [11], J. Stallings [13], and G. Baumslag [1], and for general $|m|, |n|, |p| \geq 2$ by M. P. Schützenberger [12] and by Schützenberger and Lyndon [7].

The last result is contained in the following theorem of Baumslag [2]. Suppose that $w = W(x_1, \cdots, x_n)$ is an element of the free group F freely generated by $x_1, \cdots, x_n$, that w is not a *primitive,* in other words, is not a member of a free basis of F, and that w is not a proper power, that is, $w \neq u^k$ if $k > 1$ and $u \in F$. If elements $y_1, \cdots, y_{n+1}$ satisfy the relation $W(y_1, \cdots, y_n) = y_{n+1}^m$ for some $m > 1$ and generate a free group, then the rank of this free group is at most $n - 1$.

We obtain a theorem that contains Baumslag's result:

THEOREM 1. *Let* $w = W(X(x_1, \cdots, x_n), Y(y_1, \cdots, y_m))$ $(w \neq 1)$ *be an element of the free group* F *freely generated by* $x_1, \cdots, x_n, y_1, \cdots, y_m$. *Suppose that neither* X *nor* Y *is a proper power, and set* $W(X, 1) = X^h$ and $W(1, Y) = Y^k$. *In order that elements* $u_1, \cdots, u_n, v_1, \cdots, v_m$ *satisfying the relation*

$$W(X(u_1, \cdots, u_n), Y(v_1, \cdots, v_m)) = 1$$

generate a free group of rank $n + m - 1$, *it is necessary and sufficient that at least one of the following three conditions hold.*

(i) $X(x_1, \cdots, x_n)$ *and* $Y(y_1, \cdots, y_m)$ *are both primitive.*

(ii) $X(x_1, \cdots, x_n)$ *is primitive and* k *is a multiple of* h, *or* $Y(y_1, \cdots, y_m)$ *is primitive and* h *is a multiple of* k.

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