SOME INTRICATE NONINVERTIBLE LINKS

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Let L be an oriented, ordered link tamely imbedded in the oriented 3-sphere S^3, and let μ and κ be integers such that $1 \leq \kappa < \mu$. We shall say that L is a *generalized noninvertible link with respect to the pair* (μ, κ) (or a (μ, κ)*I-link*) if it satisfies the following three conditions:

(i) L *has* μ *components;*

(ii) *each sublink with* κ *or fewer components is invertible;*

(iii) *each sublink with more than* κ *components is noninvertible.*

L is *invertible* provided it is of the same (oriented) type as its inverse. The *inverse* differs from L only in the orientation of each component.

Now (2, 1)I-links were exhibited in [6], and a $(\mu, \mu - 1)$I-link was given in [7] for each $\mu \geq 3$ (see also Figure 1). In this paper, we complete the picture by constructing a generalized noninvertible link for each pair (μ, κ) such that $1 \leq \kappa < \mu$ and $\mu \geq 3$. As an example, a (4, 2)I-link is given in Figure 4.

1. TWO PROPOSITIONS

The following two propositions clear the way for the constructive proof of our theorem in Section 3.

PROPOSITION 1. *For each integer* $\mu \geq 2$, *there exists a* $(\mu, 1)$*I-link in* S^3.

Proof. Each component of each (2, 1)I-link of [6] is of knot type 5_1. As an induction hypothesis, suppose that L is a $(\mu, 1)$I-link with $\mu \geq 2$ and that each component of L is of knot type 5_1. Let $K_{\mu+1}$ denote an oriented knot of type 5_1 in S^3 - L, and suppose that for each $\nu = 1, \cdots, \mu$ it represents an element of $\pi_1(S^3 - K_\nu)$ that cannot be mapped onto its inverse by any inversion [5] of this group. By [6], such an element of $\pi_1(S^3 - K_\nu)$ exists. In conjunction with the induction hypothesis, this means that each sublink of $L \cup K_{\mu+1}$ of two or more components is noninvertible. Hence, $L \cup K_{\mu+1}$ is a $(\mu + 1, 1)$I-link, and the conclusion follows by induction.

PROPOSITION 2. *For each integer* $\mu \geq 2$, *there exists a* $(\mu, \mu - 1)$*I-link in* S^3.

Proof. This proposition states the combined contents of [6] and [7]. However, in view of our objective in this paper of constructing generalized noninvertible links, it is convenient to give for each $\mu \geq 3$ a $(\mu, \mu - 1)$I-link different from that described in [7].

The link L of Figure 1 is assumed to have $\mu \geq 3$ components, each of which is of trivial knot type. Note that the sublink $L^* = K_2 \cup \cdots \cup K_\mu$ is a link of Brunnian type [1], so that L^* is unsplittable while each of its proper sublinks is completely splittable. Furthermore, it is easy to see that each proper sublink of L is invertible. A proof that L is noninvertible can be constructed along the lines given in [7].

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