A THEOREM ON HOMOTOPY-COMMUTATIVITY

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In [6], higher forms of homotopy-commutativity, C_n-forms, were defined for associative H-spaces. It was shown that an associative H-space admits a C_n-form if and only if its Hopf fibration $X \to E_1 \to SX$ extends to a fibration $X \to E_n \to (SX)_n$, where $(SX)_n$ denotes the n-fold James' reduced product space of the suspension of X. A C_2-form is simply a commuting homotopy for X. It is the purpose of this paper to show that the above result for $n = 2$ holds also for homotopy-commutative H-spaces that are not necessarily associative, but only homotopy-associative.

THEOREM. *Let* X *be a homotopy-associative* H-*space. Then* X *is homotopy-commutative if and only if the Hopf fibration extends to a fibration* $X \to E_2 \to (SX)_2$.

In fact, our proof will show that the "if" part of the theorem holds even *without* associativity requirements on X. We shall begin with the demonstration of this part of the theorem, then define the construction that establishes the reverse implication. We then conclude with a corollary and some illustrative applications.

Let X be an H-space, with multiplication $m: X^2 \to X$, and let $X \to E_1 \xrightarrow{p} SX$ denote the Hopf fibration for X. Since X is null-homotopic in E_1, there exists a retraction $r: \Omega SX \to X$ such that if $i: X \to \Omega SX$ denotes the usual inclusion, then ri is homotopic to the identity map of X. Furthermore, if $n: X^2 \to X$ is given by $n(x, y) = r(i(x) + i(y))$, then n is homotopic to m. (For details on these well-known facts, see [2, pp. 201-205] or [5].) Now assume that p extends to $X \to E_2 \xrightarrow{p'} (SX)_2$. Then r extends to $r': \Omega(SX)_2 \to X$. Let $j: \Omega SX \to \Omega(SX)_2$ denote the inclusion. The homotopies that are commonly used to show that the loop space of an H-space is homotopy-commutative can also serve to define a homotopy $Q': I \times (\Omega SX)^2 \to \Omega(SX)_2$ between $j(a) + j(b)$ and $j(b) + j(a)$. Let $\tilde{Q}: I \times X^2 \to X$ be the composition $r' \circ Q' \circ (1 \times i^2)$. Then $\tilde{Q}$ can be deformed to $Q: I \times X^2 \to X$, which is a commuting homotopy for m. Hence, X is homotopy-commutative.

Now let X be a homotopy-associative, homotopy-commutative H-space. As in [6, pp. 194-195], let K_n be the convex hull in R^n of the orbit of the point $(1, 2, \cdots, n)$ under permutation of the coordinates. [See [3] for a picture of K_n $(n \leq 4)$ and for verification of the following facts.] The boundary of K_n is the union of $(n - 2)$-cells that are in one-to-one correspondence with the (ℓ, m)-shuffles of the set $\{1, 2, \cdots, n\}$ $(1 \leq \ell, m \leq n - 1)$. If (A_ℓ, B_m) is such an (ℓ, m)-shuffle, then the cell of $Bd(K_n)$ corresponding to it is the image of $K_\ell \times K_m$ by a one-to-one linear map $V(A_\ell, B_m): K_\ell \times K_m \to Bd(K_n)$. There are maps $s_j: K_{n+1} \to K_n$ $(j = 1, \cdots, n + 1)$ that interact with each other and with the $V(A_\ell, B_m)$'s somewhat in the manner of degeneracy operators. We shall be concerned with K_n only for $n = 1, 2$, and 3.

We begin the construction of E_n $(n \leq 2)$ by setting $E_0 = X$ and choosing for $a_1: X \to X$ the identity map. Let $Q: I \times X^2 \to X$ and $M: I \times X^3 \to X$ be commuting and associating homotopies for X. Let

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