RANGES OF NORMAL AND SUBNORMAL OPERATORS

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1. Let T be a bounded operator on a Hilbert space H, and denote its spectrum by sp(T) and its range by R(T). (Only bounded operators will be considered.) For each set E of complex numbers, let S(T; E) be the subset of H defined by

$$S(T;E) = \bigcap_{t \in E} R(T - tI), \qquad S(T; \text{empty set}) = H. \tag{1.1}$$

Denote the interior of E by int(E) and the complement of E by C(E). Clearly, S is a decreasing function of E in the sense that $S(T; E_1) \subset S(T; E_2)$ if $E_1 \supset E_2$. Also, since $R(T - tI) = H$ whenever t does not belong to sp(T),

$$S(T; sp(T)) \subset S(T; E) \qquad \text{for each } E. \tag{1.2}$$

If T is normal and has the spectral resolution

$$T = \int z\, dK_z, \tag{1.3}$$

let K(E) denote the associated projection measure defined on the Borel sets E of the plane. We shall prove the following result.

THEOREM 1. *If* T *is normal and has the spectral resolution* (1.3), *and if* E *is any Borel set of the plane, then*

$$S(T; C(E)) \subset R(K(E)) \subset S(T; int(C(E))). \tag{1.4}$$

Consequently,

$$S(T; sp(T) - E) = R(K(E)) \text{ if } E \text{ is a closed subset of } sp(T), \tag{1.5}$$

and, in particular,

$$S(T; sp(T)) = 0. \tag{1.6}$$

To obtain (1.5) from (1.4), note that now C(E) = int(E) and hence, by (1.4), $R(K(E)) = S(T; C(E)) = S(T; sp(T) \cap C(E)) = S(T; sp(T) - E)$.

We see that if T is normal, then S(T; sp(T)) = 0 but $S(T; E) \neq 0$ whenever E is small relative to sp(T), more precisely, whenever the closure of E is a proper subset of sp(T). In case T is not normal, simple examples show that even (1.6) can be false. We need only consider an operator $T \neq 0$ for which sp(T) is the single point 0.

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