INEQUALITIES FOR CONDENSERS, HYPERBOLIC CAPACITY, AND EXTREMAL LENGTHS

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1. INTRODUCTION

In Section 3 of this paper, we give a pair of elementary estimates for the p-capacity of a condenser in Euclidean n-space, taken with respect to an arbitrary metric g. Various choices for g yield a number of useful bounds for the conformal or n-capacity (Sections 4 and 5). We use two of these bounds to derive a distortion theorem for plane quasiconformal mappings (Section 6) and to obtain sharp bounds for the hyperbolic capacity of a plane set (Section 7). In Sections 8 and 9, we employ two other bounds to study the relation between the moduli of the two families of Jordan curves that link the interior and exterior, respectively, of a torus in 3-space.

2. NOTATION

We consider sets in Euclidean n-space R^n $(n \geq 2)$ and in its one-point compactification $\overline{R}^n$ obtained by adding the point ∞ to R^n. Points in R^n are treated as vectors, and for each $x \in R^n$ we let $|x|$ denote the norm of x. For each set $E \subset \overline{R}^n$, we let ∂E, $\overline{E}$, and C(E) denote the boundary, closure, and complement of E in $\overline{R}^n$, while for $E \subset R^n$ and $k \in (0, \infty)$, we let $m_k(E)$ denote the k-dimensional Hausdorff measure of E. In particular, m_n will denote Lebesgue measure in R^n.

A *condenser* R is a domain in R^n whose complement consists of two distinguished disjoint closed sets C_0 and C_1. R is a *ring* if, in addition, C_0 and C_1 are connected. For convenience of notation, we shall always assume that $\infty \in C_1$.

Suppose that g is a function that is positive and continuous in a condenser R. Then, for $p \in (1, \infty)$, we define the *p-capacity of* R *with respect to* g as

$$(1) \qquad \operatorname{cap}_p(R, g) = \inf_u \int_R |\operatorname{grad} u|^p g^{n-p} dm_n,$$

where the infimum is taken over all functions u that are continuous in $\overline{R}^n$ and ACT (absolutely continuous in the sense of Tonelli) in R^n, with u = 0 in C_0 and u = 1 in C_1. We call any such function u an *admissible function* for R. The usual p-capacity of R [27] is then simply the p-capacity of R with respect to the function g = 1, that is,

$$(2) \qquad \operatorname{cap}_p(R) = \operatorname{cap}_p(R, 1),$$

while for the conformal or n-capacity of R [17] we have the relation

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