# AN INFINITE-DIMENSIONAL VERSION OF LIAPUNOV'S CONVEXITY THEOREM

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The classical theorem of Liapunov asserts that the range of a finite measure with values in a finite-dimensional vector space is convex and closed (see [1], [2], [3], [4]). In his later paper [5], Liapunov gives an example of an $L_1$-valued measure whose range is compact but not convex. In this note, we prove a weaker version of Liapunov's theorem, where the measure takes values in a Hilbert space and is absolutely continuous with respect to a numerical measure.

Let $(S, \mathscr{F}, \mu)$ denote a measure space, where $\mu$ is a positive, nonatomic measure with $\mu(S) = 1$, and let H denote a real Hilbert space with the inner product $(x, y)$ and norm $\|x\|$.

THEOREM. *Let* $f\colon S \to H$ *be an integrable function* (*that is,* $\int \|f\|\, d\mu < \infty$), *and let* $R = R(f)$ *be the set of all vectors of the form* $\int_E f\, d\mu$ $(E \in \mathscr{F})$. *Then* $\overline{R}$ *is convex.*

The proof is motivated by a method due to Halkin [2] who considered the finite-dimensional case only. We need several lemmas.

LEMMA 1. *Let* $\{x_1', x_2', \cdots, x_N'\}$ *be a collection of* N *vectors in* H *such that* $\sum x_i' = 0$. *Then the* $x_j'$ *can be rearranged to form a set* $\{x_1, x_2, \cdots, x_N\}$ *such that*

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 \leq \sum_{i=1}^{N} \|x_i\|^2 \qquad (1 \leq n \leq N).$$

*Proof.* We choose $x_1$ arbitrarily. Having chosen $x_1, x_2, \cdots, x_n$, we select $x_{n+1}$ to be one of the remaining vectors with the property that

$$(x_1 + x_2 + \cdots + x_n,\ x_{n+1}) \leq 0.$$

Such a choice is always possible, because

$$0 = \left( \sum_1^N x_i', \sum_1^N x_i' \right) = \left( \sum_1^n x_i, \sum_1^n x_i \right) + 2 \sum_{j=n+1}^{N} \left( \sum_1^n x_i, x_j' \right) + \left( \sum_{n+1}^{N} x_j', \sum_{n+1}^{N} x_j' \right).$$

Since the first and the last inner products are nonnegative, at least one summand in the middle term must be nonpositive. Our arrangement of the $x_j$ gives us the equations

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