THE WEAK CONTINUITY OF METRIC PROJECTIONS

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Let X be a Banach space, and let M be a closed subspace in X. Define P_M to be the metric projection (nearest-point operator, best-approximation operator) supported by M; that is, if x is an element of X, then

$$P_M(x) = \{y \in M \mid \|x - y\| = \inf_{z \in M} \|x - z\|\}.$$

M is said to be a Chebyshev subspace provided $P_M(x)$ is a singleton for each x in X.

There has been recent interest [2], [3], [7] in the continuity behavior of the metric projection P_M, especially when continuity is determined by topological conditions on the kernel $P_M^{-1}(\theta) = \{x \in X \mid P_M(x) = \theta\}$ [2]. The purpose of this paper is to establish sufficient conditions for the metric projection to be weakly continuous (that is, continuous as a mapping from the weak topology to the weak topology). The main result is Theorem 1. Theorem 2 and its corollaries are intended to simplify the hypotheses of Theorem 1. Theorem 3 is an extension of the result for the bw-topology. Two examples at the end of the paper establish the necessity of some of the hypotheses.

For the weak sequential topology, R. B. Holmes has recently proved a result [2, Theorem 11] analogous to Theorem 1.

THEOREM 1. *If* M *is a finite-dimensional Chevyshev subspace of* X *such that* $P_M^{-1}(\theta)$ *is weakly closed, then* P_M *is weakly continuous.*

Proof. Let $\{u_\alpha\}$ be a net converging weakly to u in X. We shall show that $\{P_M(u_\alpha)\}$ converges weakly to $P_M(u)$. We may assume $P_M(u) = \theta$. Let $S_M = \{x \in M \mid \|x\| = 1\}$ and $U_M = \{x \in M \mid \|x\| < 1\}$. Because S_M is weakly compact, $S_M + P_M^{-1}(\theta)$ is weakly closed. We claim that $V = P_M^{-1}(U_M)$ is weakly open. Supposing to the contrary that there is a net $\{y_\beta\}$ in $X \sim V$ that is convergent weakly to a point y in V, we have the inequality $\|P_M(y_\beta)\| \geq 1$ for each β, while $\|P_M(y)\| < 1$. Using the fact that P_M is norm-continuous (see for example [6, page 347]), we obtain for each β a number $t_\beta \in [0, 1]$ and a point $v_\beta = t_\beta y_\beta + (1 - t_\beta)y$ such that $\|P_M(v_\beta)\| = 1$, in other words, such that $\{v_\beta\} \subset S_M + P_M^{-1}(\theta) = P_M^{-1}(S_M)$. Because $\{v_\beta\}$ converges weakly to y and $S_M + P_M^{-1}(\theta)$ is weakly closed, y is an element of $X \sim V$, a contradiction. Thus V is weakly open, and since $u \in V$, we see that $\{u_\alpha\}$ is eventually in V. Hence $\{P_M(u_\alpha)\}$ is eventually in U_M, and therefore it has a norm cluster point, say z. Taking subnets if necessary, we may assume that $\{P_M(u_\alpha)\}$ converges in norm to z. For each α, let

$$d_\alpha = \inf \{\|u_\alpha - x\| \mid x \in P_M^{-1}(z)\}.$$

Then $\{d_\alpha\}$ converges to 0, since $u_\alpha + (z - P_M(u_\alpha))$ is in $P_M^{-1}(z)$ for each α, and $\{z - P_M(u_\alpha)\}$ converges in norm to θ. If for each α we choose $w_\alpha \in P_M^{-1}(z)$ so

Received October 17, 1969.

Michigan Math. J. 17 (1970).