ON TWO INVARIANT σ-ALGEBRAS FOR AN AFFINE TRANSFORMATION

Nobuo Aoki

1. INTRODUCTION

Suppose G is a compact, connected, abelian group and T (T: G → G) is an ergodic affine transformation. We shall prove that the maximal factor transformation of T with quasi-discrete spectrum is the maximal factor of T whose entropy is zero. This result was first obtained by W. Parry [5] for the case where G is metrizable. I benefitted from reading the papers by W. Parry [5] and P. Walters [6], and I am grateful to the referee for helpful suggestions.

2. PRELIMINARIES

An affine transformation T of a compact, connected, abelian group G is a transformation of the form $T(x) = aA(x)$ $(x \in G)$, where A is a continuous group automorphism of G and where $a \in G$. Such transformations T preserve Haar measure. For a compact, connected, abelian group G with Haar measure m, we consider the normalized measure space $(G, \mathcal{E}, m)$, where $\mathcal{E}$ is the completion of the σ-algebra generated by the open subsets of G (it is not a Lebesgue space, since G is nonmetrizable).

A collection $\eta = \{E_t\}$ of $\mathcal{E}$-measurable sets with the property that

$$\bigcup_t E_t = G \qquad \text{and} \qquad E_t \cap E_{t'} = \emptyset \quad (t \neq t')$$

is called an $\mathcal{E}$-*measurable partition.*

If ζ is an $\mathcal{E}$-measurable partition, we denote by $\mathscr{B}(\zeta)$ the σ-algebra generated by the members of ζ. Then $\mathscr{B}(\zeta)$ is a sub-σ-algebra of $\mathcal{E}$. Suppose $\{\zeta_\alpha\}$ is a collection of $\mathcal{E}$-measurable partitions. Then the algebra generated by $\bigcup_\alpha \mathscr{B}(\zeta_\alpha)$ consists of finite unions of sets of the form $\bigcap_{j=1}^n A_{\alpha_j}$, where $A_{\alpha_j} \in \mathscr{B}(\zeta_{\alpha_j})$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a finite subset of the collection of indices. By $\bigvee_\alpha \mathscr{B}(\zeta_\alpha)$, we denote the σ-algebra generated by $\bigcup_\alpha \mathscr{B}(\zeta_\alpha)$.

Suppose η is an $\mathcal{E}$-measurable partition of G; then H denotes the projection of G onto the factor space G_η; in other words, H maps a point of G onto the element of η to which it belongs. If $T\eta = \eta$ (mod 0), then the factor transformation T_η is induced by T, that is, $T_\eta = HTH^{-1}$.

Let $\mathcal{E}_\eta$ be the σ-algebra generated by the subsets of G_η that belong to the sub-σ-algebra $\mathscr{B}(\eta)$, and let m_η denote the measure on $\mathcal{E}_\eta$ induced by m. Then T_η is an automorphism of the factor space $(G_\eta, \mathcal{E}_\eta, m_\eta)$, and

Received June 30, 1969.

Michigan Math. J. 17 (1970).