CELLULARITY CRITERIA FOR MAPS

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In [13], D. R. McMillan gave a criterion for cellularity of a compact set A in a PL n-manifold M ($n \neq 4$). Clearly, any such criterion must say that A behaves topologically like a cell (a sphere can never be cellular), and it must also say something about the way A is embedded in M (an arc may fail to be cellular in euclidean space). This is not necessarily the case with cellularity criteria for maps.

It is known, for example, that a proper map $f: M \to N$ between topological n-manifolds ($n \geq 5$) is cellular provided for each $y \in N$ the space $f^{-1}(y)$ is cell-like (a topological property, defined below): thus there is no need for assumptions on the embeddings $f^{-1}(y) \subset M$ ($y \in N$). In the present paper, we relax the topological conditions on point-inverses in two situations: for self-maps of a PL-manifold, and for maps between topological manifolds. The general idea of the criteria is to assume that point-inverses behave like k-connected spaces (that is, have property UV^k), where k is almost n/2. Then properties of the induced map on homology, together with duality and a kind of Hurewicz theorem for UV-properties, imply that point-inverses actually behave like contractible spaces, which implies that they are cell-like and that each inclusion $f^{-1}(y) \subset M$ satisfies McMillan's criterion. Our conditions for maps are best possible codimensionally, and they are necessary as well as sufficient.

Addendum. In Section 7, we examine the case in which point-inverses have property UV^{k-1} and M has dimension 2k. In this, the critical codimension, we show that all but a finite number of point-inverses must be cellular in M (assuming M is compact and PL, and $k \neq 2$). A result of L. C. Siebenmann then implies that M is homeomorphic to the connected sum of N and a finite number of closed, (k - 1)-connected manifolds.

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Conventions. R^k is euclidean k-space, B^k is the closed unit ball in R^k, and $S^{k-1} = \partial B^k$. The symbols H_*, H^*, and $\check{H}^*$ denote singular homology, singular cohomology, and Čech cohomology, each with integer (Z) coefficients. The symbol ~ over a (co)homology symbol indicates "reduced". See [20] for a general reference on algebraic topics.

ANR's are always assumed to be metrizable. When $A \subset X$, a *neighborhood* of A in X is always understood to be an open set of X containing A. A map $f: X \to Y$ is *proper* if and only if $f^{-1}(K)$ is compact for all compact sets $K \subset Y$.

Convention on manifolds. In all statements and proofs, a *manifold* will be taken to be a connected, locally euclidean metric space. (No boundary points are allowed.)

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