EXTENSIONS TO THE DISK OF PROPERLY NESTED PLANE IMMERSIONS OF THE CIRCLE

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1. INTRODUCTION

Let $f: S \to E$ be an immersion of the oriented circle into the oriented plane. For each point Q in the complement of the image [f], the *winding number* $\omega(f, Q)$ is the topological degree of the map $t \to (f(t) - Q)$ of the circle into the punctured plane. The *tangent winding number* $\tau(f)$ is the degree of the velocity $t \to f'(t)$. If f is an interior boundary (that is, if f extends to a map of the disk D, and if this map is topologically equivalent to an analytic map), then f has *nonnegative circulation* (that is, $\omega(f, Q) \geq 0$ for all Q not in [f]). C. J. Titus showed in [4, p. 435] that the converse of this is false. If f extends to an immersion of D, then $\tau(f) = +1$. The immersion f is said to be *normal* if its image [f] lies in general position. Thus, a normal immersion f has but a finite number of selfintersections. If N is a selfintersection of [f], then $f^{-1}(N) = \{t_1, t_2\}$, $t_1 \neq t_2$, and the two vectors $f'(t_1)$ and $f'(t_2)$ are linearly independent. We call such a point a *node* of [f]. (It was called *Knotenpunkt* in [2], crossing point in [6], double point in [4], and vertex in [5].) The specification of a preferred outside starting point, as in [6, p. 281], orients each of the nodes. A normal immersion is *properly nested* if each of its nodes is a cut point of the graph (that is, decomposes [f] into two disjoint figures). See Figure 2, for example.

Titus [4] gave a simple combinatorial criterion under which a properly nested immersion of the circle extends to an immersion of the disk. The proof follows, for example, from [5, p. 60]. (A different, explicit proof, based on [1], will be included in a subsequent paper.) In this paper we offer a proof of the following theorem based on the remarkable work of S. J. Blank [3].

THEOREM 1. *A properly nested, normal immersion of the circle into the plane has at most one class of topologically equivalent extensions to an immersion of the disk.*

Proof. Recall [6, p. 281] that the tangent winding number of a normal immersion may be computed as the algebraic sum of the orientations of the nodes, plus the orientation of the outside starting point. Thus, unless [f] is a Jordan loop, $\tau(f) = 1$ implies that f has at least one negatively oriented node. Nonnegative circulation for f implies that the outside starting point and the first subsequent node must both be positive. If in addition f is properly nested, we can always find a first negative node N, preceded by a positive node M. We can isolate a region $\mathscr{R}$, enclosing the regions $\mathscr{L}$ and $\mathscr{S}$ such that $[f] \cap (\mathscr{R} \setminus \mathscr{L} \cup \mathscr{S})$ has the appearance indicated by the solid lines in Figure 3. We replace the simple arc XMNY of [f] with the arc XZY over the region $\mathscr{L}$. This new curve may be parametrized so as to produce another properly nested immersion g with $\tau(g) = 1$. Its image [g] has two fewer nodes that [f]. In Section 3, we shall apply the methods of S. Blank [3] to prove the following proposition.

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