ON THE STABLE SUSPENSION HOMOMORPHISM

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1. INTRODUCTION

The suspension homomorphism $s^q\colon \pi_i X \to \pi_{i+q} S^q X$ has been studied and exploited for a long time. Many of its properties are known, such as the fact that s^q is an isomorphism if X is k-connected and $i \leq 2k$. Let $\pi_i^s X$ denote the stable group $\pi_{i+q} S^q X$, for large values of q, and let $s = s^q$. In this paper, we prove a refinement of a theorem about the homomorphism $s\colon \pi_i X \to \pi_i^s X$ when we are in the metastable range, that is, when X is k-connected and $i \leq 3k$. It is probably the best possible theorem of this type. Our result should not be surprising, because there is much that can be done in the so-called metastable range in general. The condition of being in that range appears naturally in numerous problems. For example, the metastable range plays an important role in the imbedding theorems of [5] and [10], in the computation of the homotopy groups of S^n and O_n (see [2] and [11]), and in the E-H-P sequence of G. Whitehead [17].

Throughout this paper, C_t will denote the class of finite abelian groups whose elements have orders that divide some power of the order of $\pi_1^s \oplus \cdots \oplus \pi_t^s$, where $\pi_i^s = \pi_i^s S^0$ is the ith stable homotopy group of the sphere (C_t is the zero class for $t \leq 0$). By C, we denote the class of all finite abelian groups. For simplicity, all spaces will be finite CW-complexes with base points, which, however, will frequently be ignored.

Our main theorem follows.

THEOREM 1. *Let* X *be a* (k - 1)*-connected space.*

(a) *If* $2k \geq n + 3$, *then* $s\colon \pi_{n+k+1} SX \to \pi_{n+k}^s X$ *is* C_{n-k}*-onto.*

(b) *Let* K *denote the kernel of the map* $s\colon \pi_{n+k} X \to \pi_{n+k}^s X$. *If* $2k \geq n + 4$, *then* $s^1(K) \subseteq \pi_{n+k+1} SX$ *belongs to* C_{n-k+1}*; in other words,* $s \mid s^1(\pi_{n+k} X)$ *is a* C_{n-k+1}*-monomorphism.*

Note. It was pointed out to the author that Theorem 1 is true without any connectivity or dimension hypotheses if we replace C_{n-k} by C. This follows from theorems in homotopy theory and knowledge of the structure of Hopf algebras (see [13]). However, Theorem 1 is of value for two reasons: First, we obtain more information, because we use the classes C_{n-k} and C_{n-k+1}; and second, differential topologists would consider our proof elementary and much simpler than the proof of the general result. (Our proof may seem somewhat long, but this is due to Lemma 4, which we need to overcome a technical problem. The basic idea is really contained in Lemma 5.)

We also note that it is not always true in (a) that $\pi_{n+k} X \to \pi_{n+k}^s X$ is C-onto (consider the Eilenberg-MacLane space $X = K(\mathbb{Z}, k)$), nor is the group K in (b) always finite ($\pi_{4q-1} S^{2q} \to \pi_{4q-1}^s S^{2q}$ has infinite kernel).

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