PARACOMPACTNESS OF LOCALLY COMPACT HAUSDORFF SPACES

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A topological space is *paracompact* if it is a Hausdorff space and if every open cover has a locally finite refinement that is also an open cover.

Let X be a locally compact Hausdorff space, let $A = C(X)$ be the ring of all continuous real-valued functions on X, and let J(X) be the ideal in A consisting of all continuous functions having compact support.

THEOREM (R. Bkouche). *The space* X *is paracompact if and only if* J(X) *is a projective* A-*module*.

This theorem is a corollary of a deep result [1] of R. Bkouche. The authors heard of it through P. Samuel, who suggested that an elementary proof would be desirable.

Recall that if a space X is paracompact and $\{V_\beta\}$ is an open cover of X, then there exists a *partition of unity subordinate to* $\{V_\beta\}$; in other words, there exist continuous functions $f_\beta\colon X \to I = [0, 1]$ such that

i) for each β, $\operatorname{supp} f_\beta = \overline{\{x \in X\colon f_\beta(x) \neq 0\}} \subset V_\beta$;

ii) the family $\{\operatorname{supp} f_\beta\}$ is a locally finite cover of X;

iii) for each $x \in X$, $1 = \sum_\beta f_\beta(x)$.

An A-module M is projective [2, p. 132, Proposition 3.1] if and only if it has a projective basis, that is, if there exist elements $f_\beta \in M$ and A-homomorphisms $\phi_\beta\colon M \to A$ such that for each $g \in M$,

i) $\phi_\beta(g) = 0$ for almost all β,

ii) $g = \sum_\beta \phi_\beta(g) f_\beta$.

Also, in a locally compact Hausdorff space each compact subset K has a compact neighborhood in X, and for each such neighborhood V there exists a continuous separating function $s\colon X \to I$ that is 1 on K and 0 on $X - V$.

X *is paracompact* $\Rightarrow$ J *is projective.* Let $\{U_\alpha\}$ be a covering of X by open sets with compact closure. Since X is paracompact, there exists a locally finite refinement $\{V_\beta\}$ (of course, each $\overline{V}_\beta \subset \overline{U}_\beta$ is compact). If $\{f_\beta\}$ is a partition of unity subordinate to $\{V_\beta\}$, then each f_β has compact support, hence lies in J.

For each β, let s_β be a separating function that is 1 on the support of f_β and 0 on $X - V_\beta$. Define $\phi_\beta\colon J \to A$ by

$$\phi_\beta(g) = g s_\beta, \qquad \text{where } g \in J.$$

We claim that the f_β and ϕ_β give a projective basis of J.

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