PARACOMPACTNESS OF LOCALLY COMPACT
HAUSDORFF SPACES

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A topological space is *paracompact* if it is a Hausdorff space and if every open
cover has a locally finite refinement that is also an open cover.

Let $X$ be a locally compact Hausdorff space, let $A = C(X)$ be the ring of all con-
tinuous real-valued functions on $X$, and let $J(X)$ be the ideal in $A$ consisting of all
continuous functions having compact support.

**THEOREM (R. Bkouche).** The space $X$ is paracompact if and only if $J(X)$ is a
projective $A$-module.

This theorem is a corollary of a deep result [1] of R. Bkouche. The authors
heard of it through P. Samuel, who suggested that an elementary proof would be
desirable.

Recall that if a space $X$ is paracompact and $\{V_\beta\}$ is an open cover of $X$, then
there exists a *partition of unity subordinate to* $\{V_\beta\}$; in other words, there exist
continuous functions $f_\beta: X \to I = [0, 1]$ such that

i) for each $\beta$, $\text{supp } f_\beta = \{x \in X: f_\beta(x) \neq 0\} \subseteq V_\beta$;

ii) the family $\{\text{supp } f_\beta\}$ is a locally finite cover of $X$;

iii) for each $x \in X$, $1 = \sum f_\beta(x)$.

An $A$-module $M$ is projective [2, p. 132, Proposition 3.1] if and only if it has a
projective basis, that is, if there exist elements $f_\beta \in M$ and $A$-homomorphisms
$\phi_\beta: M \to A$ such that for each $g \in M$,

i) $\phi_\beta(g) = 0$ for almost all $\beta$,

ii) $g = \sum \phi_\beta(g) f_\beta$.

Also, in a locally compact Hausdorff space each compact subset $K$ has a com-
pact neighborhood in $X$, and for each such neighborhood $V$ there exists a continuous
separating function $s: X \to I$ that is 1 on $K$ and 0 on $X - V$.

$X$ *is paracompact* $\Rightarrow$ $J$ *is projective*. Let $\{U_\alpha\}$ be a covering of $X$ by open
sets with compact closure. Since $X$ is paracompact, there exists a locally finite
refinement $\{V_\beta\}$ (of course, each $V_\beta \subseteq U_\beta$ is compact). If $\{f_\beta\}$ is a partition
of unity subordinate to $\{V_\beta\}$, then each $f_\beta$ has compact support, hence lies in $J$.

For each $\beta$, let $s_\beta$ be a separating function that is 1 on the support of $f_\beta$ and 0
on $X - V_\beta$. Define $\phi_\beta: J \to A$ by

$$\phi_\beta(g) = gs_\beta, \quad \text{where } g \in J.$$ 

We claim that the $f_\beta$ and $\phi_\beta$ give a projective basis of $J$.

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