SCARCITY OF ORIENTATION-REVERSING PL INVOLUTIONS OF LENS SPACES

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1. INTRODUCTION

For convenience, we shall not consider the 3-sphere as a lens space. The following theorem justifies the title of this paper.

THEOREM. (i) *No lens space other than the projective* 3-*space* P_3 *admits an orientation-reversing involution.* (ii) *Up to* PL-*equivalences, there exists exactly one orientation-reversing* PL *involution of* P_3.

Part (i) is not new, but we have included it for emphasis. It follows from [5, Theorem V], and it is a special case of the result in [2]. We remark that the unique involution of Part (ii) is the one induced by the reflection of S^3 about the equator. The fixed-point set is the disjoint union of a projective plane and a point. As a corollary, we obtain the following result.

COROLLARY. *There exists no* PL *action of* $Z_2 + Z_2$ *on* S^3 *that leaves a four-point set* A *invariant (as a set) and acts freely off* A.

By a four-point set, we mean a set consisting of four distinct points. The corollary restricts PL actions of $Z_2 + Z_2$ on S^3.

Henceforth, let h denote an orientation-reversing PL involution of P_3 with fixed-point set F. It is a consequence of the parity theorem and the Lefschetz fixed-point formula that $\dim F = 0$ or $\dim F = 2$. We shall rule out the case $\dim F = 0$ in Section 2, establish the uniqueness for the case $\dim F = 2$ in Section 3, and prove the corollary in Section 4.

2. THE CASE $\dim F = 0$

2.1. We shall prove that h fixes exactly two points. Suppose h fixes $x_1, x_2, \cdots, x_k \in P_3$ and no other point. It seems to be known (and it is fairly easy to prove) that a PL involution of a finite simplicial complex becomes simplicial after a suitable subdivision. Hence we may assume h is simplicial with vertices x_i. Further, we assume that the closed stars of x_i are mutually disjoint. Let X be obtained from P_3 by removing open stars of the x_i. Then $h' = h \mid X$ is a free involution of X reversing the orientation of each boundary component of the 3-manifold X.

The Lefschetz number of h' is

$$1 - 0 + (1 - k) = 2 - k.$$

Hence $k = 2$.

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