LINDELÖF REALCOMPACTIFICATIONS

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A topological space X is called an I-space if every collection of closed sets with the countable intersection property (c.i.p.) is contained in a maximal collection of closed sets with the c.i.p. This notion was introduced by R. W. Bagley and J. D. McKnight [1]. Examples of I-spaces are the Lindelöf spaces and the countably compact spaces. In this note, we examine under what conditions the realcompactification υX of an I-space X is a Lindelöf space. We also settle a question raised by the paper of Bagley and McKnight.

We refer the reader to L. Gillman and M. Jerison [3] for such matters as the definition and the basic properties of υX, where X is a completely regular space, and for terminology. For example, a z-filter is a "filter" of zero sets of continuous, real-valued functions on X [3, page 24]. All spaces in this paper are completely regular.

LEMMA 1. *The realcompactification* υX *of* X *is a Lindelöf space if and only if every* z*-filter in* X *with the* c.i.p. *is contained in a* z*-ultrafilter with the* c.i.p.

Proof. Note that if Z is the zero set of a continuous real function f on X and $cl_{\upsilon X}$ denotes the closure operator in υX, then $cl_{\upsilon X}\, Z$ is the zero set of f^{υ}, the natural extension of f to υX [3, page 118]. Also, if Z_i $(i = 1, 2, \cdots)$ are zero sets, then

$$cl_{\upsilon X} \bigcap_i Z_i = \bigcap_i cl_{\upsilon X}\, Z_i\,.$$

Thus, the collections of zero sets of X having the c.i.p. are paired by extension with the collections of zero sets of υX having the c.i.p. Since every z-ultrafilter in υX with the c.i.p. has nonempty intersection, our lemma can be restated as follows: υX is a Lindelöf space if and only if every z-filter in υX with the c.i.p. has nonempty intersection. We have thus reduced the lemma to Problem 8H.5 of [3].

LEMMA 2. *If* X *is an* I*-space, then* υX *is a Lindelöf space.*

Proof. Let $\mathscr{F}$ be a z-filter with the c.i.p. Let $\mathscr{C}$ denote a maximal collection of closed sets with the c.i.p. containing $\mathscr{F}$. Let $\mathscr{C}'$ denote the collection of zero sets in $\mathscr{C}$. Using the maximality of $\mathscr{C}$ and an argument of the type appearing on page 30 of [3], we see that $\mathscr{C}'$ is a prime z-filter. Thus $Z(O^p) \subseteq \mathscr{C}' \subseteq Z(M^p)$ for some $p \in \beta X$, and the z-ultrafilter containing $\mathscr{C}'$ has the c.i.p. by Problem 7H.3 of [3]. By Lemma 1, υX is a Lindelöf space.

Our first theorem generalizes Theorem 2 in [1].

THEOREM 1. *A space* X *is both realcompact and an* I*-space if and only if* X *is a Lindelöf space.*

Proof. If X is a Lindelöf space, then X is realcompact, and as we remarked above, X is an I-space. The converse follows from Lemma 2. (J. E. Keesling has obtained an independent proof of Theorem 1.)

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