A NOTE ON FIBRATIONS AND CATEGORY

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Let cat X denote the Lusternik-Schnirelmann category of X as redefined by G. W. Whitehead [4], and suppose cat X is renormalized to take the value 0 on contractible spaces. Let $p: E \to B$ be a Hurewicz fibration, where B is arc-connected. Using an alternative definition of cat X that is equivalent for a large class of spaces, K. Varadarajan [3] proved the inequality

$$\text{cat } E \leq \text{cat } F + \text{cat } B + \text{cat } F \text{ cat } B, \tag{1}$$

where F denotes the fibre above some point * of B. Suppose that (B, *) is a closed cofibred pair. The purpose of this note is to prove the inequality

$$\text{cat } E \leq \text{cat } i + \text{cat } p + \text{cat } i \text{ cat } p, \tag{2}$$

where $i: F \to E$ denotes the injection and where the right-hand side is to be interpreted in the sense of the extension to maps of the (renormalized) definition of category due to Whitehead. (See [1].) Each map $f: Y \to B$ such that cat f < cat B, converted into a fibration, yields an example for which (2) is sharper than (1).

Let $\Pi^n X$ be the n-fold product of the based space X with itself, let $\Delta_X = \Delta_X^n: X \to \Pi^n X$ be the diagonal map, and let $j = j_X: T^n X \to \Pi^n X$ be the map that injects the fat wedge. Suppose that cat p = n - 1. We recall that under these conditions there exists a map $\phi: E \to T^n B$ such that

$$j_B \cdot \phi \sim \Delta_B \cdot p. \tag{3}$$

Since $\Pi^n(p): \Pi^n E \to \Pi^n B$ is a fibration and since $\Pi^n(p) \cdot \Delta_E = \Delta_B \cdot p$, the homotopy (3) may be lifted to a homotopy $\Delta_E \sim \phi': E \to \Pi^n E$, where

$$\Pi^n(p) \cdot \phi' = j_B \cdot \phi. \tag{4}$$

Now suppose that cat i = m - 1, and choose $\theta: F \to T^m E$ such that

$$j_E \cdot \theta \sim \Delta_E^m \cdot i. \tag{5}$$

Since the map $* \to B$ is a closed cofibration, it follows from [2; Theorem 12] that i is a cofibration. Hence the homotopy (5) can be extended to a homotopy $\Delta_E^m \sim \tau: E \to \Pi^m E$, where τ is such that

$$\tau \cdot i = j_E \cdot \theta. \tag{6}$$

Now Π^n is a functor that respects homotopies; hence we have the relations

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