THE HARDY CLASS OF SOME UNIVALENT FUNCTIONS AND THEIR DERIVATIVES

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1. INTRODUCTION

If $f(z) = \sum_0^\infty a_n z^n$ is a function analytic for $|z| < 1$, then $f(z)$ is said to belong to H^λ $(\lambda > 0)$ if

$$M_\lambda(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta\right)^{1/\lambda} \leq K \qquad (0 \leq r < 1),$$

where K is a constant depending on $f(z)$. We denote by H^∞ the class of analytic functions bounded for $|z| < 1$.

In this section, we list some known theorems and lemmas for reference.

THEOREM A. *If* $f(z) \in H^\lambda$ $(0 < \lambda < 1)$, *then*

$$a_n = o(n^{1/\lambda - 1}).$$

THEOREM B. *If* $f(z)$ *is univalent, then* $f(z) \in H^\lambda$, *for all* $\lambda < 1/2$.

THEOREM C. *If* $f(z)$ *is univalent, then*

$$|a_n|/n \to \alpha |a_1|$$

as $n \to \infty$, *where* $0 \leq \alpha \leq 1$.

Theorem A is in [2], Theorem C in [6, p. 104], and Theorem B is, for example, in [9, p. 214].

The Koebe function $z(1 - z)^{-2} = \sum_1^\infty nz^n$ shows that there exist univalent functions that are not in $H^{1/2}$. We have, in fact, the following result.

THEOREM D. *If* $f(z)$ *is univalent, then*

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{1/2} d\theta \Big/ \log \frac{1}{1 - r} = 2|a_1|^{1/2} \alpha^{1/2},$$

where α *is as in Theorem* C.

Theorem D is an immediate consequence of Theorems I and VI and Lemma I in [5].

According to Theorem D, univalent functions whose coefficients satisfy the relation $|a_n|/n \not\to 0$ are necessarily excluded from $H^{1/2}$. It is less obvious that there

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