A BANACH SPACE OF LOCALLY UNIVALENT FUNCTIONS

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1. INTRODUCTION

In this paper, we study a certain real Banach space $\mathscr{L}$ of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

that are holomorphic and locally univalent ($f'(z) \neq 0$) in the open unit disk $D = \{z: |z| < 1\}$. We let S denote the set of functions f(z) that are holomorphic and univalent in D with an expansion of the form (1.1). The algebraic operations in $\mathscr{L}$ (defined in Section 2 below) are not the usual pointwise operations, and the algebraic structure of $\mathscr{L}$ is of particular interest in relation to S, because local univalence in D is preserved by the addition in $\mathscr{L}$. We also study a certain closed subspace of $\mathscr{L}$, denoted by $\mathscr{L}_1$. Our spaces $\mathscr{L}$ and $\mathscr{L}_1$ are natural generalizations of a space introduced by H. Hornich [11].

The main results in this paper pertain to the metric properties of the sets $S \cap \mathscr{L}$ and $S \cap \mathscr{L}_1$. The set $S \cap \mathscr{L}$ is not compact, and it is of first category in $\mathscr{L}$. We also show that there are no isolated univalent functions in $\mathscr{L}_1$. These results contrast sharply with theorems of H. Hornich [11] and G. Piranian [16]. Hornich [10] and Piranian [16] have studied topological properties of the set of univalent functions in the space $H(\phi)$ of functions f(z) holomorphic in D, equipped with pointwise operations and a metric $\rho(f, g) = \phi(f - g)$ induced by the functional

$$\phi(f) = \sup_n |f^{(n)}(0)/n!|^{1/n}.$$

We also show that $\mathscr{L}_1$ is separable and has infinite dimension. We show that $\mathscr{K}$, the set of univalent convex functions of the form (1.1), is a closed convex subset of $\mathscr{L}_1$. A complete characterization of the extreme points of $\mathscr{K}$ is given. We determine the dual space of continuous linear functionals on $\mathscr{L}_1$. We list examples and results that indicate the relationship of $\mathscr{L}$ (as a set of functions) to the Hardy spaces H^p and to the set of functions holomorphic on D and continuous on the closure of D.

2. THE LINEAR SPACES $\mathscr{L}$ AND $\mathscr{L}_1$

Let Λ denote the class of functions that are holomorphic in the unit disk, have nonvanishing derivative, and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. When we refer to a class of functions, we shall mean the intersection of that class with Λ. For each f in Λ, define the increasing function

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