TRANSLATION IN MEASURE ALGEBRAS AND THE CORRESPONDENCE TO FOURIER TRANSFORMS VANISHING AT INFINITY

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Let G denote a locally compact (not necessarily abelian) group and M(G) the collection of finite regular Borel measures on G. The set M(G) is a semisimple Banach algebra with identity under convolution *. It can be identified with the dual space of $C_0(G)$, the space of continuous complex-valued functions on G that vanish at infinity, with the sup-norm. The group G has a left-invariant regular Borel measure dm(x) that is unique up to a constant and is called the left Haar measure of G. Let $C^B(G)$ denote the space of bounded continuous functions on G. For each $x \in G$, we define on $C^B(G)$ the left-translation operator by the relation

$$L(x)f(y) = f(x^{-1}y) \qquad (f \in C^B(G)).$$

We say that $f \in C^B(G)$ is right uniformly continuous if $L(x_\alpha)f \xrightarrow{\alpha} L(x)f$ uniformly, whenever $x_\alpha \xrightarrow{\alpha} x$. Let $C^B_{ru}(G)$ denote the subspace of $C^B(G)$ of right uniformly continuous functions. For $\mu \in M(G)$, define $L(x)\mu \in M(G)$ by the condition

$$\int_G f(t)\,dL(x)\mu(t) = \int_G L(x^{-1})f(t)\,d\mu(t),$$

where $f \in C_0(G)$. We wish to study for which $\mu \in M(G)$ the map $x \mapsto L(x)\mu$ is continuous from G into M(G), where M(G) will be equipped with an L(x)-invariant metric topology. In particular, we shall characterize $M_0(G)$, the algebra of measures whose Fourier transform vanishes at infinity.

Let $A \subset C^B_{ru}(G)$ be a linear subspace with sufficiently many elements to separate the points of M(G); in other words, if $\mu \in M(G)$ and if

$$\int_G f(t)\,d\mu(t) = 0$$

for all $f \in A$, then $\mu = 0$. We are then able to pair A and M(G) by the relation

$$\langle f, \mu \rangle = \int_G f(t)\,d\mu(t) \qquad (f \in A;\ \mu \in M(G)).$$

Let $\sigma(A, M(G))$ denote the weak topology on A induced by this pairing. Suppose A can be written as $\bigcup_{k=1}^\infty A_k$, where each A_k is a subset of A that is L(x)-invariant for all $x \in G$ and where each A_k is $\sigma(A, M(G))$-bounded. Note that A_k is

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