THE BIRATIONALITY OF CUBIC SURFACES OVER A GIVEN FIELD

H. P. F. Swinnerton-Dyer

Let V be a nonsingular cubic surface in 3-dimensional projective space, and assume that V is defined over a given algebraic number field k. It is well known that over the complex numbers any such V is birationally equivalent to a projective plane. The problem of finding necessary and sufficient conditions for V to be birationally equivalent to a projective plane over k was first raised by B. Segre; and partial answers to it have been given by Segre [3] and J. I. Manin [1]. In this paper, I use Segre's methods to give a complete answer to the problem; for the reader's convenience, I have developed the argument *ab initio,* rather than quote intermediate results from [3].

Following Segre, we denote by S_n any subset of the 27 straight lines on V that satisfies the conditions below:

(i) S_n consists of n lines, no two of which meet.

(ii) If S_n contains a line L, then S_n also contains all the conjugates of L over k.

Because of (i), we have that $n \leq 6$. We call an S_n *irreducible* if it consists of a line and its conjugates over k. We shall prove the following result.

THEOREM. *A necessary and sufficient condition that* V *should be birationally equivalent to a projective plane over* k *is that* V *should contain a point defined over* k *and that* V *should have at least one* S_2, S_3, *or* S_6.

The condition that V should contain a point defined over k (which is clearly necessary) can be put into an equivalent form in which it can be more easily checked, if the other condition is satisfied. It follows from the construction below that if V has an S_2, it automatically contains points defined over k. Again, if V has an S_3 or an S_6, then it contains points defined over k if and only if it contains points defined over each $\mathfrak{p}$-adic field, where $\mathfrak{p}$ runs through all the primes of k; for a proof of this result, which was first discovered by Châtelet, see [1] or [4].

Let $\bar{k}$ denote the algebraic closure of k. In what follows, we have to distinguish between the geometric properties of V, which are defined over $\bar{k}$ or the complex numbers, and the arithmetic properties of V, which are defined over k. In the language of schemes, this is just the distinction between $V \otimes_k \bar{k}$ and V. For geometric purposes, we can obtain a model for V as follows. Choose six skew lines on V; each of these is an exceptional curve of the first kind and can therefore be blown down into a point. By blowing down all six of these lines, we birationally transform V (over $\bar{k}$) into a plane containing six distinguished points $P_1, \cdots, P_6$. No three of these points are collinear, and they do not all lie on a conic. The 27 lines on V correspond to the 6 points P_i, the 15 lines $P_i P_j$, and the 6 conics each of which passes through five of the P_i; from this correspondence, their incidence relations can easily be read off. The plane sections of V correspond to the cubic curves passing

Received April 13, 1970.

Michigan Math. J. 17 (1970).