REPRESENTATIONS OF INTEGRAL RELATION ALGEBRAS

Ralph McKenzie

The main object of this note is to prove the following theorem.

The class of relation algebras that possess representations over a group is not finitely axiomatizable relative to the class of representable, integral relation algebras.

Previously it was not known whether the two classes are distinct. (The question had been stated as an open problem in [2].) To prove the theorem, we shall define and study an intermediate class of relation algebras. Roger Lyndon suggested an appropriate generic name for these algebras: *permutational.* An algebra will be called permutational if one of its representations admits a transitive group of automorphisms. Probably the most important unsolved problem related to our work is the question whether every representable, integral relation algebra is permutational. We shall strengthen some results of R. C. Lydon's paper [5] to obtain a negative solution of this problem under the assumption that there exists a finite projective plane whose order is not a power of a prime integer.

The final section contains the presentation of a nonrepresentable relation algebra having the smallest possible size.

The author is indebted to the referee for the correction of several inaccuracies.

1. PRELIMINARIES

A *relation algebra* is a universal algebra of the type $\mathfrak{A} = \langle A, +, \cdot, -, ;, ^{\smile}, 1' \rangle$ that satisfies certain postulates due to Tarski (see for example [3, Definition 4.1]): $\langle A, +, \cdot, - \rangle$ is a Boolean algebra, the formulae

$$(x ; y) ; z = x ; (y ; z) \quad \text{and} \quad x ; 1' = 1' ; x = x$$

hold for all $x, y, z \in A$, and the formulae

$$(x ; y) \cdot z = 0, \quad (x^{\smile} ; z) \cdot y = 0, \quad (z ; y^{\smile}) \cdot x = 0$$

are equivalent for all $x, y, z \in A$. We use the symbols 0 and 1 to denote the Boolean null and unit element of $\mathfrak{A}$. A relation algebra is *representable* if it is isomorphic to an algebra $\mathfrak{D} = \langle D, \cup, \cap, \sim, | , ^{\smile}, I \rangle$, where, for some set X, $\langle D, \cup, \cap, \sim \rangle$ is a Boolean algebra of subsets of $X \times X$ (whose unit set is not necessarily equal to $X \times X$), where I is the identity relation on X, and where for any R and S belonging to D, $R | S$ is the relative product of R and S and $R^{\smile}$ is the converse of R. A representation of $\mathfrak{A}$ over the set X is a pair $\langle \mathfrak{D}, \phi \rangle$, where $\mathfrak{D}$

Received May 4, 1970.

This research was supported by an NSF Cooperative Graduate Fellowship (1964-66). The results constitute part of the author's doctoral dissertation [8], prepared under the supervision of Professor J. D. Monk.

Michigan Math. J. 17 (1970).