ON THE KRULL-SCHMIDT THEOREM FOR INTEGRAL GROUP REPRESENTATIONS OF RANK 1

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0. INTRODUCTION

Let R be a Dedekind domain with quotient field K (char K = 0). Let G be a finite group. An RG-module M is a finitely generated, torsionfree left R-module on which G acts from the left by R-homomorphisms

$$g: M \to M, \quad m \mapsto gm \qquad (g \in G,\ m \in M).$$

We assume M to be imbedded in $KM = K \otimes_R M$, and we define $\operatorname{rk} M = \operatorname{Dim}_K KM$.

In [5], I. Reiner constructs counterexamples to the Krull-Schmidt Theorem for R' G-modules for the case where R' is a ring of algebraic numbers in some algebraic number field K (integral at all places $\mathfrak{p}$ dividing $|G|$), where the order of G is not a power of a prime, and where G contains a normal subgroup whose index is a prime. He points out that this method does not work for simple groups.

In this note, we provide easy counterexamples for groups G and rings R for which the ideal $|G|R$ is not primary in R; in other words, $|G|R$ is contained in at least two different prime ideals. Thus we furnish counterexamples in the setting of [5], if G is not a p-group or if G is a p-group, but p decomposes in K.

We consider RG-modules of rank 1 and find conditions for an arbitrary RG-module M to be of the form M' + M" with rk M' = 1. Our starting point is the following observation:

For each $U \leq G$, let $\mathbb{Z}[G/U]$ denote the $\mathbb{Z}G$-module that is spanned as $\mathbb{Z}$-module by the left cosets $gU \in G/U$ with the obvious G-action. Then the trivial $\mathbb{Z}G$-module $\mathbb{Z} = \mathbb{Z}[G/G]$ is a direct summand in $\mathbb{Z}[G/U]$ if and only if U = G (in fact, every module $\mathbb{Z}[G/U]$ is indecomposable). But $\mathbb{Z}$ is a direct summand in

$$\bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p]$$

(where G_p is a p-Sylow subgroup in G), because the map $T_U: \mathbb{Z}[G/U] \twoheadrightarrow \mathbb{Z}$ $(gU \mapsto 1)$ has no right inverse if $U \neq G$, whereas the map

$$\bigoplus_{p \mid |G|} T_{G_p}: \bigoplus_{p \mid |G|} \mathbb{Z}[G/G_p] \to \mathbb{Z}$$

has a right inverse.

Our results obviously do not apply in the situation where R is a discrete valuation ring, and they even seem to support the conjecture that the Krull-Schmidt

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